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NUMERICAL SOLUTION OF A MIXED PROBLEM FOR A TWO-DIMENSIONAL SYSTEM OF SAINT-VENANT EQUATIONS

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Abstract

The work is devoted to the construction and study of a numerical method for solving the Saint-Venant equation. These equations are of great practical importance in modern hydraulic engineering and are suitable for describing natural processes in the atmosphere, rivers, and oceans, as well as for modeling tides. Questions of formulation of mixed problems for the two-dimensional system of Saint-Venant equations, are studied. A new upwind difference scheme of splitting in spatial directions is constructed for solving the mixed problem of the two-dimensional Saint-Venant equation, which describes flows without turbulent diffusion components. The stability of the difference scheme concerning energy norms is established. The results of numerical experiments for model problems are presented, including a numerical simulation of water flow in the Ugam River. The numerical calculation is based on the use of the two-point sweep method.

Keywords: two-dimensional system of Saint-Venant equations, upwind difference scheme of splitting on directions, stability.

Аңдатпа

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СЕН-ВЕНАН ТЕҢДЕУЛЕРІНІҢ ЕКІ ӨЛШЕМДІ ЖҮЙЕСІ ҮШІН АРАЛАС ЕСЕПТІҢ САНДЫҚ ШЕШІМІ

Жұмыс Сент-Венан теңдеуін шешудің сандық әдісін құруға және зерттеуге арналған. Бұл теңдеулер қазіргі гидротехникада үлкен практикалық маңызға ие және атмосферадағы, өзендер мен мұхиттардағы табиғи процестерді сипаттау үшін, сондай-ақ толқындарды модельдеу үшін қолайлы. Сен-Венан теңдеулерінің екі өлшемді жүйесіне аралас есептерді шығару сұрақтары зерттеледі. Турбулентті диффузиялық құраушыларсыз ағындарды сипаттайтын екі өлшемді Сен-Венан теңдеуінің аралас есебін шешу үшін кеңістіктік бағыттағы бөлудің жаңа желге қарсы айырымы схемасы құрастырылған. Энергия нормаларына қатысты айырмашылық схемасының тұрақтылығы белгіленеді. Угам өзеніндегі су ағынын сандық модельдеуді қоса алғанда, модельдік есептер бойынша сандық тәжірибелердің нәтижелері ұсынылған. Сандық есептеу екінүктелік прогонка әдісін қолдануға негізделген.

Түйін сөздер: Сен-Венан теңдеулерінің екі өлшемді жүйесі, бағыттар бойынша бөлудің ағынға қарсы айырымдық сұлбасы, орнықтылық.

Аннотация

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ЧИСЛЕННОЕ РЕШЕНИЕ СМЕШАННОЙ ЗАДАЧИ ДЛЯ ДВУМЕРНОЙ СИСТЕМЫ УРАВНЕНИЙ СЕН-ВЕНАНА

Работа посвящена построению и исследованию численного метода решения уравнения Сен-Венана. Эти уравнения имеют важное прикладное значение в современной гидротехнике и пригодны для описания природных процессов в атмосфере, реках и океанах, а также для моделирования приливов и отливов. Исследуются вопросы постановки смешанных задач для двумерной системы уравнений Сен-Венана. Построена новая противопоточная разностная схема расщепления по пространственным направлениям для решения смешанной задачи двумерного уравнения Сен-Венана, описывающая течения без компонентов турбулентной диффузии. Установлена устойчивость разностной схемы по энергетическим нормам. Приведены результаты

численных экспериментов для модельных задач, в том числе проведено численное моделирование течения воды в реке Угам. Численный расчет основан на использовании метода двухточечной прогонки.

Ключевые слова: двумерная система уравнений Сен-Венана, противопоточная разностная схема расщепления по направлениям, устойчивость.

1. Introduction

For the numerical solution of various problems for the system of Saint-Venant equations, scientists have proposed a number of methods [1] - [8]. In [9] approaches with separation of gaps and methods of end-to-end counting are described pretty detailed. The authors of this book focus on Godunov-type methods based on exact and approximate solutions to the Riemann problem of arbitrary gap decay. It is known that the essential properties of difference schemes are their conservativeness [8], [10] - [12], well-balanced [7], [13], [14], possibility of end-to-end calculation, choice of explicit/implicit scheme [15], approximation order, convergence, stability. When choosing a difference scheme an important property is its stability. In the case of conditionally stable schemes, a restriction is imposed on the choice of the integration step on time. A slight violation of the stability condition can lead to an increase in error and non-physical solutions. For hyperbolic systems, the time step is determined from the Courant condition.

2. Problem statement

The system of Saint-Venant equations describing flows without turbulent diffusion components has the following form [16]:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + g\dfrac{\partial H}{\partial x} = \phi_1, \\ \dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + g\dfrac{\partial H}{\partial y} = \phi_2, \\ \quad \dfrac{\partial H}{\partial t} + \dfrac{\partial Hu}{\partial x} + \dfrac{\partial Hv}{\partial y} = 0. \end{cases} \tag{1}$$

where

$$\phi_1 = lv - g\frac{\sqrt{u^2+v^2}}{C_s^2 H}u + \frac{\tau_x}{\rho H} + \frac{\partial h}{\partial x},$$
$$\phi_2 = -lu - g\frac{\sqrt{u^2+v^2}}{C_s^2 H}v + \frac{\tau_y}{\rho H} + \frac{\partial h}{\partial y},$$

x, y – directions of orthogonal coordinate axes, t – time, u, v – speed components, H –water depth, ρ – water density, g – acceleration of free fall, τ_x, τ_y – components of wind stress on the water surface, C_s – Shezi coefficient, l – Coriolis parameter.

(1) can be reduced to the following system of linear hyperbolic equations:

$$\frac{\partial V}{\partial t} + A\frac{\partial V}{\partial x} + B\frac{\partial V}{\partial y} = F, \tag{2}$$

where

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},\ A = \begin{pmatrix} u & 0 & c \\ 0 & u & 0 \\ c & 0 & u \end{pmatrix},\ B = \begin{pmatrix} v & 0 & 0 \\ 0 & v & c \\ 0 & c & v \end{pmatrix},\ F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

$u,\ v,\ c$ are given functions.

$$f_1=(v_1)\left\{u_x+l-g^2\frac{2(u)^2+v^2}{2C_s^2(c)^2U}\right\}+(v_2)\left\{u_y+l-g^2\frac{uv}{2C_s^2(c)^2U}\right\}+\frac{v_3}{2}\left\{2c_x+g^2\frac{uU}{C_s^2(c)^3}-\frac{g\tau_x}{4\rho(c)^3}\right\},$$

$$f_2=(v_1)\left\{v_x-l-g^2\frac{uv}{2C_s^2(c)^2U}\right\}+(v_2)\left\{v_y+g^2\frac{2(v)^2+(u)^2}{2C_s^2(c)^2U}\right\}+\frac{v_3}{2}\left\{2c_y+g^2\frac{vU}{C_s^2(c)^3}-\frac{g\tau_y}{4\rho(c)^3}\right\},$$

$$f_3=u_xv_3/2+2c_xv_1+v_yv_3/2+2c_yv_2.$$

$$U=\sqrt{(u)^2+(v)^2}$$

The matrix A depending on the value u is represented as the sum of two respectively A^+ non-negatively defined and A^- non-positively defined matrices:

$$A=A^++A^-,$$

It should be noted that there are 5 possible cases for u:

$$u<-c;\qquad -c\le u<0;\qquad u=0;\qquad 0<u<c;\qquad c\le u.$$

Similarly, the matrix B, depending on the value .v. is represented as the sum of two respectively B^+ non-negatively defined and B^- non-positively defined matrices:

$$B=B^++B^-,$$

There are also 5 possible cases for v:

$$v<-c;\qquad -c\le v<0;\qquad v=0;\qquad 0<v<c;\qquad c\le v.$$

In this case, system (2) will take the following record form:

$$\frac{\partial V}{\partial t}+\left(A^++A^-\right)\frac{\partial V}{\partial x}+\left(B^++B^-\right)\frac{\partial V}{\partial y}=F,$$

Exactly this kind of matrix splitting A and B into the sum of two sign -defined matrices, respectively on A^+, B^+ - non-negatively defined part and on A^-, B^- - non-positively defined part gives us the opportunity to build an upwind difference scheme.

It should be noted that in [17] a numerical calculation of Lyapunov-stable solutions of the one-dimensional Saint-Venant equation was carried out using an upwind implicit difference scheme on the example of the Big Almaty Canal.

In domain $Q=\{(t,x,y):\ 0\le t<+\infty,\ 0\le x\le X,\ 0\le y\le Y\}$ let us build a difference grid with steps Δt on direction t, $\Delta x=X/J$ on direction x and $\Delta y=Y/L$ on direction y. Here $[a]$ denotes the integer part of a real number a. J, L - some integers. Difference grid steps Δx, Δy we select it in such a way that the equalities are fulfilled $J\Delta x=X$ и $L\Delta x=Y$. This means that the difference grid completely covers the areas Q. Denote $t^\kappa=\kappa\Delta t,\ \kappa=\overline{0,K}$; $x_j=j\Delta x,\ j=\overline{0,J}$ and $y_l=l\Delta y,\ l=\overline{0,L}$. The nodal points of the difference grid (meaning the intersection of straight lines $t=t^\kappa$, $x=x_j$ and $y=y_l$) we denote by $\left(t^\kappa,x_j,y_l\right)$. The set of nodal points of the difference grid is denoted by Q_h, where

$$Q_h=\left\{\left(t^\kappa,x_j,y_l\right):\ \kappa=\overline{0,K};\ j=\overline{0,J};\ l=\overline{0,L}\right\}.$$

And the values of the numerical solution at the nodal points are denoted by

$$V_{jl}^\kappa=V\left(t^\kappa,x_j,y_l\right),\quad \kappa=\overline{0,K};\ j=\overline{0,J};\ l=\overline{0,L}.$$

To find a numerical solution to a mixed problem over a difference grid Q_h, we propose the following upwind implicit difference scheme

$$I.\quad \frac{\bar V_{jl}^\kappa-V_{jl}^\kappa}{\Delta t}+\left(B^+\right)_{jl}^\kappa\frac{V_{jl}^\kappa-V_{jl-1}^\kappa}{\Delta y}+\left(B^-\right)_{jl}^\kappa\frac{V_{jl+1}^\kappa-V_{jl}^\kappa}{\Delta y}=0,\qquad \kappa=\overline{0,K-1};\ j=\overline{1,J-1};\ l=\overline{1,L-1}.$$

$$II.\quad \frac{V_{jl}^{\kappa+1}-\bar V_{jl}^\kappa}{\Delta t}+\left(A^+\right)_{jl}^\kappa\frac{V_{jl}^{\kappa+1}-V_{j-1l}^{\kappa+1}}{\Delta x}+\left(A^-\right)_{jl}^\kappa\frac{V_{j+1l}^{\kappa+1}-V_{jl}^{\kappa+1}}{\Delta x}=F_{jl}^\kappa,\qquad \kappa=\overline{0,K-1};\ j=\overline{1,J-1};\ l=\overline{1,L-1}.$$

In the difference scheme (I)-(II), the value of $\bar{V}_{jl}^{\kappa}$ is an auxiliary intermediate value for determining the value of $V_{jl}^{\kappa+1}$ through V_{jl}^{κ}. In the algorithm of finding a numerical solution, the values of $\bar{V}_{jl}^{\kappa}$ are first determined from (I), and then substituted into (II), thereby the values of $V_{jl}^{\kappa+1}$ are determined.

3. Calculation experiment

The numerical experiment was carried out on a computer with the following technical characteristics: Intel(R) Core(TM) i7-8700, 16Gb, Windows 10, and PTC Mathcad Prime 7.

The Ugam mountain river was chosen as a computational experiment. Ugam has a fast current, below the Yugansk forestry its speed is 1.6 m/s. The width in the lower reaches (above the village of Charvak) is 23 m, the depth is 70 cm, the bottom soil is viscous. The bed of the Ugam is empty.

Let us introduce the parameters of the difference grid: time $t = 5\,s$, time step $\Delta t = 0.03$, length of the river $X = 200\ m$, length step $\Delta x = 0.5$, width of the river $Y = 23\ m$, width step $\Delta y = 0.5$.

Let us calculate the l Coriolis parameter by the formula

$$l = 2\Omega \sin\varphi$$

where Ω is an angular velocity of the Earth's rotation around the axis; φ — is a geographical latitude of the location. When crossing the equator, the Coriolis parameter and, accordingly, the Coriolis force change the sign.

The angular velocity of the Earth's rotation is:

$$\Omega = \frac{2\pi}{T} = 7.2921 \cdot 10^{-5}\ s^{-1}$$

where T — sidereal period of the Earth rotation around the axis, equal to one sidereal day (23 hours 56 minutes, 4,0905 seconds).

Ugam is located at geographical latitude $41^o 37' 46''$, respectively

$$\varphi = 41^o 37' 46'' \approx 41{,}629444^o$$

Then l is Coriolis parameter for the Ugam river:

$$l = 2\Omega \sin\varphi = 2 \cdot 7.2921 \cdot 10^{-5} \cdot \sin(41.629444) = -10.3469869 \cdot 10^{-5}$$

Resistance coefficient $C_s -$ can be determined by the formula of N.N. Pavlovsky:

$$C_s = \frac{1}{n} R^m$$

Where n is the roughness coefficient characterizing the condition of the channel surface for the case of sewer pipes is taken in the range (0,012…0,015); for other cases, information is given in the literature [18]. R is a hydraulic radius. The value of the hydraulic radius varies depending on the size and shape of the cross-section of the channel, for open channels of large width it is assumed to be equal to the average depth of the stream. Accordingly, the hydraulic radius for the Ugam River is 0.6 m.

m is an indicator of the degree depending on the value of the roughness coefficient and the hydraulic radius:

$$m = 2.5\sqrt{n} - 0.13 - 0.75\sqrt{R}(\sqrt{n} - 0.1)$$

This formula is recommended for values $R \in (3,..,5)$. With large hydraulic radii or other values of roughness coefficients, the use of Pavlovsky's formula in hydraulic calculations of riverbeds leads to significant errors. At the value $m = 1/6$ the Shezi formula is reduced to the Manning formula.

The roughness coefficient for the Ugam River is equal to $n = 0.025$. Accordingly, the exponent for the Ugam river will be equal to

$$m = 2.5\sqrt{0.025} - 0.13 - 0.75\sqrt{0.6}(\sqrt{0.025} - 0.1) = 0.23152359236$$

Shezi coefficient for the Ugam River

$$C_s = \frac{1}{n} R^m = \frac{1}{0.025} \cdot 0.6^{0.23152359236} = 35.5383100387$$

Normal wind pressure τ_x, τ_y the obstacle in the first approximation is determined by the formula

$$\tau = 0.5 \cdot \rho \cdot v^2$$

where v is a wind speed m/s, ρ is an air density kg/m^3, depending on its humidity, temperature and atmospheric pressure, 0,5 is a coefficient of resistance (flow). The Anglo-Saxons use a coefficient equal to 0,75, i.e. they receive data 1,5 times higher, but the order of magnitude, of course, is the same.

In table 1, wind pressure depending on the speed and wind strength on the obstacle. Calculation for air density 1,2 kg/m^3

Table 1. Wind load (in the first approximation)

		Wind speed in m/s								
		1 m/s	*5 m/s*	*10 m/s*	*15 m/s*	*20 m/s*	*25 m/s*	*30 m/s*	*40 m/s*	*50 m/s*
Pressure, Pa = N/m²		*0.60*	*15*	*60*	*135*	*240*	*375*	*540*	*960*	*1500*
object 1m×1m	*force per 1 m², N*	*0.60*	*15.00*	*60.00*	*135.00*	*240.00*	*375.00*	*540.00*	*960.00*	*1500.00*
object 2m×2m	*force per 4 m², N*	*2.40*	*60.00*	*240.00*	*540.00*	*960.00*	*1500.00*	*2160.00*	*3840.00*	*6000.00*
object 1m×1m	*force per 1 m², N*	*0.06*	*1.53*	*6.12*	*13.78*	*24.49*	*38.27*	*55.10*	*97.96*	*153.06*
object 2m×2m	*force per 4 m², N*	*0.24*	*6.12*	*24.49*	*55.10*	*97.96*	*153.06*	*220.41*	*391.84*	*612.24*

Ugam has a fast flow, which corresponds to the inequality $c \le u$. Then the matrix is representable as the sum of two matrices of the form:

$$A = A^+ + A^-, \quad A^+ = \begin{pmatrix} u & 0 & c \\ 0 & u & 0 \\ c & 0 & u \end{pmatrix}, \quad A^- = 0;$$

We decompose the matrices A^+, A^- as follows:

$$A^+ = PZ\Lambda_{A^+}Z^*P, \ A^- = PZ\Lambda_{A^-}Z^*P,$$
$$\Lambda_{A^+} = diag(u, \ u+c, \ u-c), \ \Lambda_{A^-} = diag(0, \ 0, \ 0),$$

then

$$A = A^+ + A^- = PZ\Lambda_{A^+}Z^*P + PZ\Lambda_{A^-}Z^*P = PZ\Lambda_A Z^*P,$$
$$\Lambda_A = \Lambda_{A^+} + \Lambda_{A^-} = diag(u, \ u+c, \ u-c).$$

Also for the speed of the river corresponds to the inequality $0 < v < c$. Then the matrix B is representable as the sum of two matrices

$$B = B^+ + B^-, \quad B^+ = \begin{pmatrix} v & 0 & 0 \\ 0 & \frac{v+c}{2} & \frac{v+c}{2} \\ 0 & \frac{v+c}{2} & \frac{v+c}{2} \end{pmatrix}, \quad B^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{v-c}{2} & \frac{c-v}{2} \\ 0 & \frac{c-v}{2} & \frac{v-c}{2} \end{pmatrix};$$

We decompose the matrices B^+, B^- as follows:

$$B^+ = PZ\Lambda_{B^+}Z^*P, \ B^- = PZ\Lambda_{B^-}Z^*P,$$
$$\Lambda_{B^+} = diag(v, \ v+c, \ 0), \ \Lambda_{B^-} = diag(0, \ 0, \ v-c),$$

then

$$B = B^+ + B^- = PZ\Lambda_{B^+}Z^*P + PZ\Lambda_{B^-}Z^*P = PZ\Lambda_B Z^*P,$$
$$\Lambda_B = \Lambda_{B^+} + \Lambda_{B^-} = diag(v,\ v+c,\ v-c).$$

Let us consider in more detail the difference scheme of splitting in directions for this case:

Splitting by direction y:

If $\left(0 < v_{jl}^{\kappa} < c_{jl}^{\kappa}\right)$, then as the first step of splitting, consider the difference scheme

$$\bar{V}_{jl}^{\kappa} = V_{jl}^{\kappa} - r_y\left(B^+\right)_{jl}^{\kappa}\left(V_{jl}^{\kappa} - V_{jl-1}^{\kappa}\right) - r_y\left(B^-\right)_{jl}^{\kappa}\left(V_{jl+1}^{\kappa} - V_{jl}^{\kappa}\right),\quad \kappa = \overline{0, K-1};\ j = \overline{1, J-1};\ l = \overline{1, L-1}.$$

We substitute the matrix $B^{\pm}$ expansions into and multiply the resulting equality on the left by the matrix Z^*P. As a result, scheme will have the following form:

$$\left(Z^*P\right)_{jl}^{\kappa}\bar{V}_{jl}^{\kappa} = \left(Z^*P\right)_{jl}^{\kappa}V_{jl}^{\kappa} - r_y\left(\Lambda_{B^+}Z^*P\right)_{jl}^{\kappa}\left(V_{jl}^{\kappa} - V_{jl-1}^{\kappa}\right) - r_y\left(\Lambda_{B^-}Z^*P\right)_{jl}^{\kappa}\left(V_{jl+1}^{\kappa} - V_{jl}^{\kappa}\right).$$

Let us rewrite this scheme in terms of vector functions $T_{jl}^{\kappa} = \left(Z^*P\right)_{jl}^{\kappa}V_{jl}^{\kappa}$ and $\bar{T}_{jl}^{\kappa} = \left(Z^*P\right)_{jl}^{\kappa}\bar{V}_{jl}^{\kappa}$:

$$\bar{T}_{jl}^{\kappa} = T_{jl}^{\kappa} - r_y\left(\Lambda_{B^+}\right)_{jl}^{\kappa}\left(T_{jl}^{\kappa} - T_{jl-1}^{\kappa}\right) - r_y\left(\Lambda_{B^-}\right)_{jl}^{\kappa}\left(T_{jl+1}^{\kappa} - T_{jl}^{\kappa}\right).$$

The component notation of this schema looks like this:

$$\begin{cases} \left(\bar{t}_1\right)_{jl}^{\kappa} = \left(t_1\right)_{jl}^{\kappa} - r_y v_{jl}^{\kappa}\left(\left(t_2\right)_{jl}^{\kappa} - \left(t_2\right)_{jl-1}^{\kappa}\right),\ l = \overline{1, L}; \\ \left(\bar{t}_2\right)_{jl}^{\kappa} = \left(t_2\right)_{jl}^{\kappa} - r_y\left(v_{jl}^{\kappa} + c_{jl}^{\kappa}\right)\left(\left(t_2\right)_{jl}^{\kappa} - \left(t_2\right)_{jl-1}^{\kappa}\right),\ l = \overline{1, L}; \\ \left(\bar{t}_3\right)_{jl}^{\kappa} = \left(t_3\right)_{jl}^{\kappa} + r_y\left(v_{jl}^{\kappa} - c_{jl}^{\kappa}\right)\left(\left(t_3\right)_{jl+1}^{\kappa} - \left(t_3\right)_{jl}^{\kappa}\right),\ l = \overline{0, L-1}. \end{cases}$$

As boundary conditions, consider the following conditions:

On the left boundary for $l = 0$, it is required to set two boundary conditions,

$$\left(v_1\right)_{j0}^{\kappa} = 0,\ \left(v_2\right)_{j0}^{\kappa} + \left(v_3\right)_{j0}^{\kappa} = 0 \Rightarrow \left(t_1\right)_{j0}^{\kappa} = 0,\ \left(t_2\right)_{j0}^{\kappa} = 0,$$

And on the right boundary for $l = L$ it is required to set one boundary condition

$$\left(v_2\right)_{jL}^{\kappa} - \left(v_3\right)_{jL}^{\kappa} = 0 \Rightarrow \left(t_3\right)_{jL}^{\kappa} = 0.$$

Note that from the equalities:

$$V = P\tilde{W} = PZT;$$
$$\tilde{w}_1 = v_2,\ \tilde{w}_2 = v_1,\ \ \tilde{w}_3 = v_3;$$
$$t_1 = \tilde{w}_2,\ t_2 = \frac{1}{\sqrt{2}}\left(\tilde{w}_1 + \tilde{w}_3\right),\ t_3 = \frac{1}{\sqrt{2}}\left(\tilde{w}_1 - \tilde{w}_3\right)$$

for $l = 0$, taking into account the boundary conditions, the validity of the following inequality for the boundary quadratic form follows:

$$-\left(\left(\Lambda_B\right)_{jl}^{\kappa}T_{jl}^{\kappa}, T_{jl}^{\kappa}\right)\Big|_{l=0} = \left(-|v|t_1^2 - |v+c|t_2^2 + |v-c|t_3^2\right)_{j0}^{\kappa} \geq 0.$$

Similarly, on the right boundary in y for $l = L$, taking into account the boundary conditions, it follows that the following inequality holds true for the boundary quadratic form:

$$\left(\left(\Lambda_B\right)_{jl}^{\kappa}T_{jl}^{\kappa}, T_{jl}^{\kappa}\right)\Big|_{l=L} = \left(|v|t_1^2 + |v+c|t_2^2 - |v-c|t_3^2\right)_{jL}^{\kappa} \geq 0.$$

Splitting by direction x:

If $\left(c_{jl}^{\kappa} \leq u_{jl}^{\kappa}\right)$, then as the second step of splitting, consider the difference scheme

$$V_{jl}^{\kappa+1} = \bar{V}_{jl}^{\kappa} - r_x \left(A^{+}\right)_{jl}^{\kappa} \left(V_{jl}^{\kappa+1} - V_{j-1l}^{\kappa+1}\right), \quad \kappa = 0, K-1; \; j = 1, J; \; l = 1, L-1.$$

We substitute the matrix $A^{\pm}$ expansions into and multiply the resulting equality on the left by the matrix $Z^{*}P$. As a result, scheme will have the following form:

$$\left(Z^{*}P\right)_{jl}^{\kappa} V_{jl}^{\kappa+1} = \left(Z^{*}P\right)_{jl}^{\kappa} \bar{V}_{jl}^{\kappa} - r_x \left(\Lambda_{A^{+}} Z^{*}P\right)_{jl}^{\kappa} \left(V_{jl}^{\kappa+1} - V_{j-1l}^{\kappa+1}\right).$$

Let's rewrite this scheme in terms of vector functions $T_{jl}^{\kappa} = \left(Z^{*}P\right)_{jl}^{\kappa} V_{jl}^{\kappa}$ and $\bar{T}_{jl}^{\kappa} = \left(Z^{*}P\right)_{jl}^{\kappa} \bar{V}_{jl}^{\kappa}$:

$$T_{jl}^{\kappa+1} = \bar{T}_{jl}^{\kappa} - r_x \left(\Lambda_{A^{+}}\right)_{jl}^{\kappa} \left(T_{jl}^{\kappa+1} - T_{j-1l}^{\kappa+1}\right)$$

The component notation of this schema looks like this:

$$\begin{cases} \left(t_1\right)_{jl}^{\kappa+1} = \left(\bar{t}_1\right)_{jl}^{\kappa} - r_x u_{jl}^{\kappa} \left(\left(t_1\right)_{jl}^{\kappa+1} - \left(t_1\right)_{j-1l}^{\kappa+1}\right), \quad j = \overline{1, J}; \\ \left(t_2\right)_{jl}^{\kappa+1} = \left(\bar{t}_2\right)_{jl}^{\kappa} - r_x \left(u_{jl}^{\kappa} + c_{jl}^{\kappa}\right) \left(\left(t_2\right)_{jl}^{\kappa+1} - \left(t_2\right)_{j-1l}^{\kappa+1}\right), \quad j = \overline{1, J}; \\ \left(t_3\right)_{jl}^{\kappa+1} = \left(\bar{t}_3\right)_{jl}^{\kappa} - r_x \left(u_{jl}^{\kappa} - c_{jl}^{\kappa}\right) \left(\left(t_3\right)_{jl}^{\kappa+1} - \left(t_3\right)_{j-1l}^{\kappa+1}\right), \quad j = \overline{1, J}. \end{cases}$$

As boundary conditions, consider the following conditions:
On the left boundary for $j = 0$, it is required to set three boundary conditions,

$$\left(v_2\right)_{0l}^{\kappa} = 0, \; \left(v_1\right)_{0l}^{\kappa} + \left(v_3\right)_{0l}^{\kappa} = 0, \; \left(v_1\right)_{0l}^{\kappa} - \left(v_3\right)_{0l}^{\kappa} = 0 \Rightarrow \left(t_1\right)_{0l}^{\kappa} = 0, \; \left(t_2\right)_{0l}^{\kappa} = 0, \; \left(t_2\right)_{0l}^{\kappa} = 0.$$

and on the right boundary for $j = J$ there is no need to set boundary conditions.

Note that from the equalities:

$$V = P\tilde{W} = PZT;$$
$$\tilde{w}_1 = v_1, \; \tilde{w}_2 = -v_2, \;\; \tilde{w}_3 = v_3;$$
$$t_1 = \tilde{w}_2, \; t_2 = \frac{1}{\sqrt{2}}\left(\tilde{w}_1 + \tilde{w}_3\right), \; t_3 = \frac{1}{\sqrt{2}}\left(\tilde{w}_1 - \tilde{w}_3\right)$$

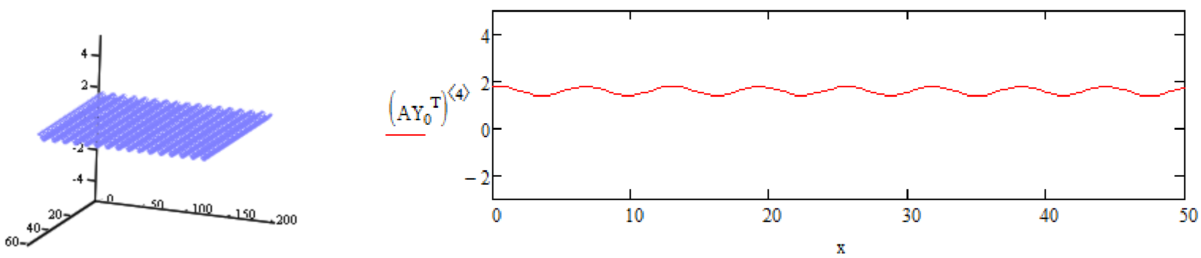
for $j = 0$, taking into account the boundary conditions, the validity of the following inequality for the boundary quadratic form follows:

$$-\left(\left(\Lambda_A\right)_{jl}^{\kappa} T_{jl}^{\kappa}, T_{jl}^{\kappa}\right)\Big|_{j=0} = \left(-|u| t_1^2 - |u+c| t_2^2 - |u-c| t_3^2\right)_{0l}^{\kappa} \geq 0.$$

Similarly, on the right boundary in y for $l = L$, taking into account the boundary conditions, it follows that the following inequality holds true for the boundary quadratic form:

$$\left(\left(\Lambda_A\right)_{jl}^{\kappa} T_{jl}^{\kappa}, T_{jl}^{\kappa}\right)\Big|_{j=J} = \left(|u| t_1^2 + |u+c| t_2^2 + |u-c| t_3^2\right)_{Jl}^{\kappa} \geq 0.$$

During the computational experiment, the following result was obtained:

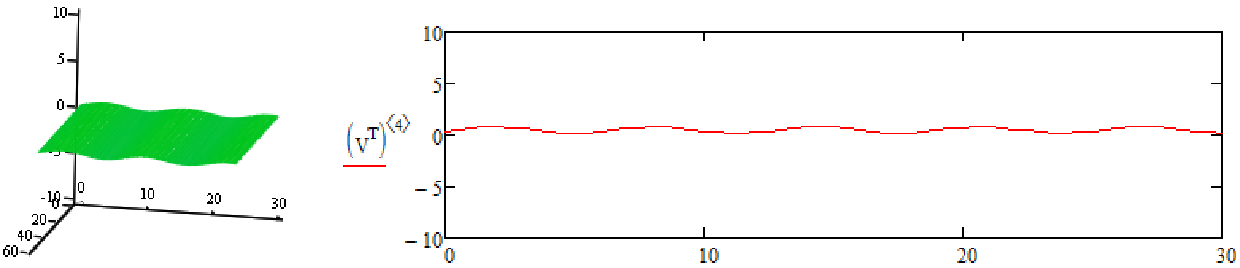


a) *b)*

Figure 1. $u-$ *velocity in the x direction at a fixed y*

Table 2. Numerical solution $u-$ velocity in the x direction at a fixed y

№	j = 1	j = 2	j = 4	j = 6	j = 8	j = 10
k=1	1.7993659589	1.7673311277	1.5982297381	1.4307559192	1.4188843276	1.57352965
k=2	1.7993658153	1.7673309631	1.5982295543	1.4307557151	1.4188841207	1.5735294606
k=3	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587
k=4	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587
k=5	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587
k=6	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587
k=7	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587
k=8	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587
k=9	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587
k=10	1.7993658154	1.767330964	1.5982295561	1.4307557163	1.4188841198	1.5735294587

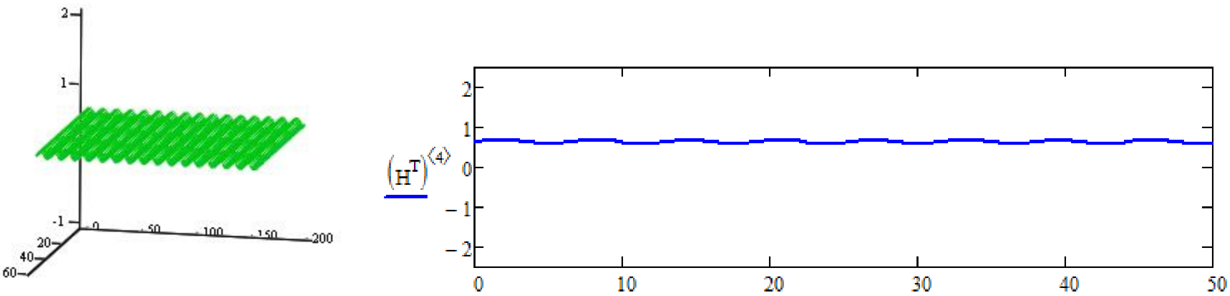


a) b)

Figure 2. $v-$ velocity in the y direction at a fixed x

Table 3. Numerical value $v-$ is a velocity in the y direction

№	j = 1	j = 2	j = 4	j = 6	j = 8	j = 10
k=1	0.5645359964	0.6970959796	0.796807474	0.6236355456	0.3367936667	0.200002938
k=2	0.5645359964	0.6970959796	0.796807474	0.6236355456	0.3367936667	0.200002938
k=3	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378
k=4	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378
k=5	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378
k=6	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378
k=7	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378
k=8	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378
k=9	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378
k=10	0.5645359977	0.6970959808	0.7968074738	0.623635544	0.336793665	0.2000029378



a) b)

Figure 3. $H-$ is a water depth in the x direction at a fixed y

Table 4. Numerical value $H-$ is a water depth

№	j = 1	j = 2	j = 4	j = 6	j = 8	j = 10
k=1	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=2	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=3	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=4	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=5	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=6	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=7	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=8	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=9	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287
k=10	0.6478523293	0.6707570423	0.6882481578	0.658014842	0.6094303071	0.5869171287

For the experiment, a plot of 200 meters long and 23 meters wide was taken. Figure 1 shows a graph of the speed change in the direction of the river length. Figure 2 shows a graph of the speed change in the direction of the river width. Figure 3 shows a graph of changes in the water height level in the direction of the Ugam River length.

4. Conclusion

In this manuscript we have considered a two-dimensional system of Saint-Venant equations. The two-dimensional system of equations takes into account many parameters affecting the flow behavior, for instance such parameters as ρ – water density, g – acceleration of free fall, φ – geographical latitude of the location, n – the roughness coefficient of the bed bottom, v – speed of wind, ρ_{air} – air density kg/m^3, which in turn depends on its humidity, temperature and atmospheric pressure, R – hydraulic radius. For the numerical solution of a mixed problem for a two-dimensional system of Saint-Venant equations, an upwind difference splitting scheme in directions was proposed. A computational experiment for the Ugam River was also carried out and described in detail.

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References:

1 Khoperskov A., Khrapov S. A numerical simulation of the shallow water flow on a complex topography // Numerical Simulations in Engineering and Science / Ed. by Srinivasa Rao. — IntechOpen, 2018. — P. 237–254. DOI: 10.5772/intechopen.71026

2 Kovyrkina O. A., Ostapenko V. V. Primeneniye skhemy KABARE dlya chislennykh resheniy melkoy vody // Marchuk Scientific Readings -2017. 2017. P. 434-440. (in Russian). DOI: https://doi.org/10.7868/S0044466917070092

3 Kovyrkina O. A., Ostapenko V. V. O postroyenii kombinirovannykh raznostnykh skhem povyshennoy tochnosti // Reports of the Academy of Sciences. Vol. 478. 2018. P. 517-522. (in Russian). DOI: https://doi.org/10.31857/S0869-56524856691-696

4 Zyuzina N. A., Ostapenko V. V. O raspade neustoychivykh sil'nykh razryvov pri approksimatsii skhemoy KABARE skalyarnogo zakona sokhraneniya s vypuklym potokom // Journal of Computational Mathematics and Mathematical Physics. 2018. Vol. 58, № 6. P. 988-1012. (in Russian). DOI: https://doi.org/10.7868/S004446691806011X

5 Shokin Yu. I. Rychkov A. D., Khakimzyanov G. S., Chubarov L. B. A combined computational algorithm for solving the problem of long surface waves runup on the shore // Russian Journal of Numerical Analysis and Mathematical Modelling. 2016. Vol. 31, no. 4. P. 217–227. https://www.researchgate.net/publication/ 311718600_A_combined_computational_algorithm_for_solving_the_problem_of_long_surface_waves_runup_on_the_shore

6 Shokin YU. I., Beyzel' S. A., Rychkov A. D., Chubarov L. B. Chislennoye modelirovaniye nakata voln tsunami na poberezh'ye s ispol'zovaniyem krupnykh chastits // Math modeling. 2015. Vol. 27, № 1. P. 99-112. (in Russian) https://www.elibrary.ru/item.asp?id=23421467

7 Yelizarova T. G., Bulatov O. V. Chislennyy algoritm resheniya regulyarnoy melkoy vody na nestrukturirovannykh setkakh // Preprints of the Institute of Applied Mathematics. M.V. Keldysh RAS. 2014. № 21. P. 1-27. (in Russian) https://keldysh.ru/papers/2014/prep2014_21.pdf

8 Khrapov S. S., Khoperskov A. V., Kuzmin N. M. and others. Chislennaya skhema dlya modelirovaniya dinamiki reaktsiy vod na osnove kombinirovannogo podkhoda SPH–TVD // Computational methods and programming. 2011. Vol. 12, № 2. P. 282-297. (in Russian) https://num-meth.ru/index.php/journal/article/view/462

9 Kulikovsky A. G., Pogorelov N. V., Semenov A. Yu. Matematicheskiye voprosy chislennogo resheniya giperbolicheskikh sistem uravneniy. M. : Physmatlit, 2001. – 607 p. (in Russian) https://www.rfbr.ru/rffi/ru/ books/o_163

10 Popov Yu.P., Samarski A.A. Polnost'yu konservativnyye raznostnyye skhemy // Journal of Comp. Math. and Math. Physics. 1969. Vol. 9, № 4. P. 953–958. (in Russian) http://www.mathnet.ru /links/b81c13a093bc319 7ead0f9c0ec45de0f /zvmmf7105.pdf

11 Skiba Y. N. Finite-difference mass and total energy conserving schemes for shallow-water equations // Rus. Meteorology and Hydrology. 1995. Vol. 2. P. 55–65. https://www.semanticscholar.org/paper/Finite-difference-mass-and-total-energy-conserving-Skiba/da0adddb534dc0de00c64fe50aa46fb2955864ab

12 Burguete J., Garc´ıa-Navarro P., Murillo J. Numerical boundary conditions for globally mass conservative methods to solve the shallow-water equations and applied to river flow // International journal for numerical methods in fluids. 2006. Vol. 51, no. 6. P. 585–615. https://doi.org/10.1002/fld.1127

13 Berthon C., Chalons C. A fully well-balanced, positive and entropy-satisfying Godunov-type method for the shallow-water equations // Mathematics of Computation. — 2015. — Vol. 85, no. 299. — P. 1281–1307 DOI: 10.1090/mcom3045

14 Castro M. J., Ortega S., Par´es C. Well-balanced methods for the shallow water equations in spherical coordinates // Computers & Fluids. 2017. Vol. 157. P. 196-207. https://doi.org/10.1016/j.compfluid.2017.08.035

15 Sukhinov A. I., Chistyakov A. Ye., Shishenya A. V., Timofeyev Ye. F. Predskazatel'noye modelirovaniye realizovannykh gidrofizicheskikh protsessov na mnogoprotsessornoy sisteme s ispol'zovaniyem yavnykh skhem // Math modeling. 2018. Vol. 30, № 3. P. 83–100. (in Russian) https://www.libnauka.ru/journal/matematicheskoe-modelirovanie/matematicheskoe-modelirovanie-2018-30-3/predskazatelnoe-modelirovanie-pribrezhnykh-gidrofizicheskikh-protsessov-na-mnogoprotsessornoy-sisteme-s-ispolzovaniem-yavnykh-skhem-matematicheskoe-modelirovanie/

16 Tran Gia Lich and Le Kim Luat. Boundary conditions for the two-dimensional Saint-Venant equation system // Appl. Math. Modelling. 1992, Vol. 16. P. 498 - 502. https://doi.org/10.1016/0307-904X(92)90044-4

17 Aloev R.D., Berdyshev A.S., Akbarova A., Baishemirov Zh. Development of an algorithm for calculating stable solutions of the Saint-Venant equation using an upwind implicit difference scheme // Eastern-European Journal of Enterprise Technologies 2021. Vol. 4, No.112, P. 47-56. https://doi.org/10.15587/1729-4061.2021.239148

18 Kiselev P.G. Spravochnik po gidravlicheskim raschetam M.: Energiya. 1972. P. 312. (in Russian) http://books.totalarch.com/handbook_of_hydraulic_calculations