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WEIGHTED ESTIMATES FOR DISCRETE ITERATED HARDY OPERATOR

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Abstract

One of the most intensively studied topics in the theory of Hardy inequalities in recent years has been the estimation of iterated operators. Since these inequalities have found their application in the study of boundedness from a weighted Lebesgue space to a local Morrie-type space, as well as in the study of weighted bilinear Hardy type inequalities. The discrete case of iterated inequalities is an open problem. An inequality involving an iteration of a discrete Hardy operator is traditionally considered difficult to evaluate because it contains three independent weight sequences and three parameters in their different ratios. The aim of this paper is to obtain the necessary and sufficient conditions for the implementation of a discrete iterative Hardy inequality.

Keywords: Inequalities, iterated operator, weight, discrete Lebesgue space.

Аңдатпа

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ДИСКРЕТТІ ХАРДИ ОПЕРАТОР ИТЕРАЦИЯСЫНЫҢ САЛМАҚТЫ БАҒАЛАУЫ

Соңғы жылдары Харди теңсіздіктері теориясында ең қарқынды зерттелген тақырыптардың бірі итерациялық операторларды бағалау болды. Себебі бұл теңсіздіктердің Лебег салмақты кеңістігінен Морри типті локальді кеңістігіне шенелімділікті зерттеуде, сондай-ақ Харди типті салмақты бисызықты теңсіздіктерді зерттеуде қолданысы табылды. Итерациялық теңсіздіктердің дискретті жағдайы ашық есеп болып табылады. Әдетте, дискретті Харди операторының итерациясымен болатын теңсіздіктерді бағалау қиын деп саналады, Өйткені оның құрамында үш тәуелсіз салмақты тізбек және әр түрлі қатынастағы үш параметр бар. Бұл жұмыстың мақсаты дискретті итерациялық Харди теңсіздіктерінің орындалуының қажетті және жеткілікті шарттарын алу болып табылады.

Түйін сөздер: Теңсіздіктер, итерациялық оператор, салмақ, дискретті Лебег кеңістігі.

Аннотация

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ВЕСОВАЯ ОЦЕНКА ДИСКРЕТНОЙ ИТЕРАЦИИ ОПЕРАТОРА ХАРДИ

Одной из наиболее интенсивно изучаемых тем в теории неравенств Харди в последние годы была оценка итерационных операторов. Поскольку эти неравенства нашли свое применение при изучении ограниченности от весового пространства Лебега в локальное пространство типа Морри, а также при изучении весовых билинейных неравенств типа Харди. Дискретный случай итерационных неравенств является открытой задачей. Неравенство, включающее итерацию дискретного оператора Харди, традиционно считается трудным для оценки, поскольку оно содержит три независимые весовые последовательности и три параметра в их разных соотношениях. Целью данной работы является получение необходимых и достаточных условий для выполнения дискретного итерационного неравенства Харди.

Ключевые слова: Неравенства, итерационный оператор, вес, дискретное пространство Лебега.

1. Introduction

Let $0 < r < \infty$ and $0 < p \le q < \infty$. Let $\{u_i\}_{i=1}^{\infty}$, $\{v_i\}_{i=1}^{\infty}$, $\{w_i\}_{i=1}^{\infty}$ be weights, i.e. non-negative sequences of real numbers. Denote by $l_{p,v}$ the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\| f \|_{l_{p,v}} := \left(\sum_{i=1}^{\infty} v_i \, | f_i |^p \right)^{\frac{1}{p}} < \infty .$$

We consider the following inequalities

$$\left(\sum_{i=1}^{\infty} u_i \left(\sum_{k=1}^{i} \left| \sum_{j=k}^{i} f_j \right|^r w_k \right)^{\frac{q}{r}} \right)^{\frac{1}{q}} \le C_1 \left(\sum_{i=1}^{\infty} v_i \, | f_i |^p \right)^{\frac{1}{p}}, \tag{1}$$

$$\left(\sum_{i=1}^{\infty} u_i \left(\sum_{k=i}^{\infty} \left| \sum_{j=i}^{k} f_j \right|^r w_k \right)^{\frac{q}{r}} \right)^{\frac{1}{q}} \le C_2 \left(\sum_{i=1}^{\infty} v_i \, | f_i |^p \right)^{\frac{1}{p}}, \tag{2}$$

for all sequences $f \in l_{p,v}$.

The aim of the this work is to obtain necessary and sufficient conditions for the fulfillment of the discrete iterated Hardy inequalities (1) and (2) for cases: $0 < p \le 1$, $0 < p \le \min\{q, r\} < \infty$; $0 < r < 1 < p \le q < \infty$; $1 \le p \le \min\{q, r\} < \infty$; $0 < r < p$, $1 < p \le q < \infty$; $0 < r < p = 1 \le q < \infty$.

2. Literature review

In recent years, one of the most intensively studied topics in the theory of Hardy inequalities has been the evaluation of iterated operators. The reason for this was the publication of the work of V.I. Burenkov and R. Oinarov [1] in which the use of iterated inequalities was necessary to prove the boundedness of Hardy type operators from a weighted Lebesgue space to a local Morrey type space. In addition, iterated Hardy inequalities have a wide application in evaluation of weighted bilinear Hardy type inequalities, which was obtained in the work of A.L. Bernardis and P.O. Salvador [2].

An inequality involving an iteration of discrete, continuous Hardy operator is considered difficult to estimate since it has three independent weights and three parameters with different ratios. Nevertheless, many papers are devoted to this type of inequality. Characterization of continuous iterated Hardy inequalities were obtained in works [3-9]. Compared to the continuous case the discrete analogue of the Hardy iterated inequality is studied very little (see [10, 11]).

3. Auxiliary statements

The notations $A << B$ and $B << A$ mean that there exists a constant $c > 0$ depending only on parameters p, r, q, such that the inequalities $A \le cB$ and $B \le cA$ are fulfilled, respectively. We write $A \approx B$ if a two-way estimation $A << B << A$ holds.

To prove the main results, we will need the following well-known equivalences.

Lemma 1. Let $\alpha > 0$ and $1 \le n < N \le \infty$. Then for all sequences $a \ge 0$ the following equivalences are satisfied

$$\left(\sum_{k=n}^{N} a_k \right)^{\alpha} \approx \sum_{k=n}^{N} \left(\sum_{i=n}^{k} a_i \right)^{\alpha - 1} a_k , \tag{3}$$

$$\left(\sum_{k=n}^{N} a_k\right)^{\alpha} \approx \sum_{k=n}^{N}\left(\sum_{i=k}^{N} a_i\right)^{\alpha-1} a_k\,, \tag{4}$$

Let

$$A_{z,\infty}^{+} = \sup_{z\le k\le\infty}\left(\sum_{j=k}^{\infty} w_j\right)^{\frac{1}{r}} v_k^{-\frac{1}{p}}, \qquad A_{1,z}^{-} = \sup_{1\le k\le z}\left(\sum_{j=1}^{k} w_j\right)^{\frac{1}{r}} v_k^{-\frac{1}{p}},$$

$$B_{z,\infty}^{+} = \left(\sum_{k=z}^{\infty}\left(\sum_{j=k}^{\infty} w_j\right)^{\frac{p}{p-r}}\left(\sum_{j=z}^{k} v_j^{1-p'}\right)^{\frac{p(r-1)}{p-r}} v_k^{1-p'}\right)^{\frac{p-r}{pr}},$$

$$B_{1,z}^{-} = \left(\sum_{k=1}^{z}\left(\sum_{j=1}^{k} w_j\right)^{\frac{p}{p-r}}\left(\sum_{j=k}^{z} v_j^{1-p'}\right)^{\frac{p(r-1)}{p-r}} v_k^{1-p'}\right)^{\frac{p-r}{pr}},$$

$$D_{z,\infty}^{+} = \sup_{k\in N,\,k\ge z}\left(\sum_{j=k}^{\infty} w_j\right)^{\frac{1}{r}}\left(\sum_{j=1}^{k} v_j^{1-p'}\right)^{\frac{1}{p'}}, \qquad D_{1,z}^{-} = \sup_{k\in N,\,k\le z}\left(\sum_{j=1}^{k} w_j\right)^{\frac{1}{r}}\left(\sum_{j=k}^{\infty} v_j^{1-p'}\right)^{\frac{1}{p'}},$$

$$E_{z,\infty}^{+} = \left(\sum_{k=z}^{\infty} w_k\left(\sum_{j=k}^{\infty} w_j\right)^{\frac{r}{p-r}}\left(\sum_{j=z}^{k} v_j^{1-p'}\right)^{\frac{r(p-1)}{p-r}}\right)^{\frac{p-r}{pr}},$$

$$E_{1,z}^{-} = \left(\sum_{k=1}^{z} w_k\left(\sum_{j=1}^{k} w_j\right)^{\frac{r}{p-r}}\left(\sum_{j=k}^{z} v_j^{1-p'}\right)^{\frac{r(p-1)}{p-r}}\right)^{\frac{p-r}{pr}},$$

$$F_{z,\infty}^{+} = \left(\sum_{k=z}^{\infty} w_k\left(\sum_{j=k}^{\infty} w_j\right)^{\frac{r}{1-r}} \max_{z\le j\le k} v_j^{\frac{r}{1-r}}\right)^{\frac{p-r}{pr}}, \qquad F_{1,z}^{-} = \left(\sum_{k=z}^{\infty} w_k\left(\sum_{j=1}^{k} w_j\right)^{\frac{r}{1-r}} \max_{k\le j\le z} v_j^{\frac{r}{1-r}}\right)^{\frac{p-r}{pr}},$$

$$J_{z,\infty}^{+} = \sup_{f\ge 0}\frac{\left(\sum_{k=z}^{\infty} w_k\left(\sum_{j=z}^{k} f_j\right)^{r}\right)^{\frac{1}{r}}}{\left(\sum_{i=z}^{\infty} v_i f_i^{\,p}\right)^{\frac{1}{p}}}, \qquad J_{1,z}^{-} = \sup_{f\ge 0}\frac{\left(\sum_{k=1}^{z} w_k\left(\sum_{j=k}^{z} f_j\right)^{r}\right)^{\frac{1}{r}}}{\left(\sum_{i=1}^{z} v_i f_i^{\,p}\right)^{\frac{1}{p}}}$$

$$U_z^{+} = \left(\sum_{i=1}^{z} u_i\right)^{\frac{1}{q}}, \qquad U_z^{-} = \left(\sum_{i=z}^{\infty} u_i\right)^{\frac{1}{q}},$$

where $z\in N$.

According to the well-known weighted Hardy inequalities, we can formulate the estimation of quantities $J^{+}_{1,\infty}$ and $J^{-}_{1,\infty}$ for different ratios of parameters r, p as following lemma [12, 13].

Lemma 2. (i) If $0<p\le 1$, $p\le r<\infty$, then $J^{\mp}_{1,\infty}\approx A^{\mp}_{1,\infty}$;

(ii) If $0<r<1<p<\infty$, then $J^{\mp}_{1,\infty}\approx B^{\mp}_{1,\infty}$;

(iii) If $1\le p\le r<\infty$, then $J^{\mp}_{1,\infty}\approx D^{\mp}_{1,\infty}$;

(iv) If $0<r<p$, $1<p<\infty$, then $J^{\mp}_{1,\infty}\approx E^{\mp}_{1,\infty}$;

(v) If $0<r<p=1$, then $J^{\mp}_{1,\infty}\approx F^{\mp}_{1,\infty}$.

In the proof of necessity, a test sequence is used, and in the proof of sufficiency, the method of splitting into "packages" the sequence of values of the iterated Hardy operator at each point is used, which makes it convenient to estimate the sums by packages, thanks to which the main results are achieved. Various classical inequalities as well as Hardy's weighted inequalities are used in the evaluation.

4. Main results

Our main results read as follows:

Theorem 1. The inequality (1) holds if and only if

(i) $G_1=\sup\limits_{z\in N}U^{-}_{z}A^{-}_{1,z}<\infty$ for $0<p\le 1$, $0<p\le\min\{q,r\}<\infty$;

(ii) $G_2=\sup\limits_{z\in N}U^{-}_{z}B^{-}_{1,z}<\infty$ for $0<r<1<p\le q<\infty$;

(iii) $G_3=\sup\limits_{z\in N}U^{-}_{z}D^{-}_{1,z}<\infty$ for $1\le p\le\min\{q,r\}<\infty$;

(iv) $G_4=\sup\limits_{z\in N}U^{-}_{z}E^{-}_{1,z}<\infty$ for $0<r<p$, $1<p\le q<\infty$;

(v) $G_5=\sup\limits_{z\in N}U^{-}_{z}F^{-}_{1,z}<\infty$ for $0<r<p=1\le q<\infty$.

Moreover, $G_i\approx C_1$ $\left(i=\overline{1,5}\right)$. Here, C_1 is the smallest constant that satisfies the inequality (1).

Theorem 2. The inequality (2) holds if and only if

(i) $H_1=\sup\limits_{z\in N}U^{+}_{z}A^{+}_{z,\infty}<\infty$ for $0<p\le 1$, $0<p\le\min\{q,r\}<\infty$;

(ii) $H_2=\sup\limits_{z\in N}U^{+}_{z}B^{+}_{z,\infty}<\infty$ for $0<r<1<p\le q<\infty$;

(iii) $H_3=\sup\limits_{z\in N}U^{+}_{z}D^{+}_{z,\infty}<\infty$ for $1\le p\le\min\{q,r\}<\infty$;

(iv) $H_4=\sup\limits_{z\in N}U^{+}_{z}E^{+}_{z,\infty}<\infty$ for $0<r<p$, $1<p\le q<\infty$;

(v) $H_5=\sup\limits_{z\in N}U^{+}_{z}F^{+}_{z,\infty}<\infty$ for $0<r<p=1\le q<\infty$.

Moreover, $H_i\approx C_2$ $\left(i=\overline{1,5}\right)$, where C_2 is the smallest constant that satisfies the inequality (2).

The proof of the Theorem 2 is analogous to the proof of Theorem 1, and requires some obvious changes, so we will only prove Theorem 1.

Proof of Theorem 1. *Necessity.* Assume that the inequality (1) holds for all $f\in l_{p,v}$. We have to show that $G_i<\infty$ $\left(i=\overline{1,5}\right)$ for all given cases.

Let $z\in N$ and $0\le f\in l_{p,v}$, i.e. f is the sequence of non-negative numbers that satisfies $\sum\limits_{i=1}^{\infty}v_i f_i^{\,p}<\infty$.

We choose a test sequence $f_z=\left\{f_{z,j}\right\}_{j=1}^{\infty}$ defined as follows:

$$f_{z,j}=\begin{cases} f_j, & 1\le j\le z;\\ 0, & j>z.\end{cases}$$

By substituting the test sequence in the left-hand side of the inequality (1), we obtain its estimation from below

$$\left(\sum_{i=1}^{\infty}u_i\left(\sum_{k=1}^{i}\left|\sum_{j=k}^{i}f_{z,j}\right|^r w_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}=\left(\sum_{i=1}^{z-1}u_i\left(\sum_{k=1}^{i}\left|\sum_{j=k}^{i}f_{z,j}\right|^r w_k\right)^{\frac{q}{r}}+\sum_{i=z}^{\infty}u_i\left(\sum_{k=1}^{i}\left|\sum_{j=k}^{i}f_{z,j}\right|^r w_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\ge$$

$$\ge\left(\sum_{i=z}^{\infty}u_i\left(\sum_{k=1}^{z}\left|\sum_{j=k}^{z}f_j\right|^r w_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}=\left(\sum_{i=z}^{\infty}u_i\right)^{\frac{1}{q}}\left(\sum_{k=1}^{z}\left|\sum_{j=k}^{z}f_j\right|^r w_k\right)^{\frac{1}{r}}.$$

If we use this estimate and substitute the test sequence, then the inequality (1) will be as follows:

$$\left(\sum_{i=z}^{\infty}u_i\right)^{\frac{1}{q}}\left(\sum_{k=1}^{z}\left|\sum_{j=k}^{z}f_j\right|^r w_k\right)^{\frac{1}{r}}\le\left(\sum_{i=1}^{\infty}u_i\left(\sum_{k=1}^{i}\left|\sum_{j=k}^{i}f_{z,j}\right|^r w_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\le C_1\left(\sum_{i=1}^{z}v_i f_i^{\,p}\right)^{\frac{1}{p}}.$$

By dividing both parts of the last inequality by $\left(\sum_{i=1}^{z}v_i f_i^{\,p}\right)^{\frac{1}{p}}>0$ and taking supremum over all $f\ge 0$, we obtain that $C_1\ge U_z^- J_{1,z}^-$. If we use Lemma 2 for this inequality, we get

(i) for $0<p\le 1$, $0<p\le\min\{q,r\}<\infty$: $U_z^- J_{1,z}^-\approx U_z^- A_{1,z}^-\le C_1$;

(ii) for $0<r<1<p\le q<\infty$: $U_z^- J_{1,z}^-\approx U_z^- B_{1,z}^-\le C_1$;

(iii) for $1\le p\le\min\{q,r\}<\infty$: $U_z^- J_{1,z}^-\approx U_z^- D_{1,z}^-\le C_1$;

(iv) for $0<r<p$, $1<p\le q<\infty$: $U_z^- J_{1,z}^-\approx U_z^- E_{1,z}^-\le C_1$;

(v) for $0<r<p=1\le q<\infty$: $U_z^- J_{1,z}^-\approx U_z^- F_{1,z}^-\le C_1$.

Taking supremum over $z\in N$, we obtain that the value $G_i<\infty$ $\left(i=\overline{1,5}\right)$ is finite for all cases and

$$G_i \ll C_1 \;\left(i=\overline{1,5}\right). \tag{5}$$

Sufficiency. Let $G_i<\infty$ $\left(i=\overline{1,5}\right)$ for all five cases. We have to show the fulfillment of the inequality (1) for all $f\in l_{p,v}$. Since the inequality

$$\left(\sum_{i=1}^{\infty}u_i\left(\sum_{k=1}^{i}\left|\sum_{j=k}^{i}f_j\right|^r w_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}\le\left(\sum_{i=1}^{\infty}u_i\left(\sum_{k=1}^{i}\left(\sum_{j=k}^{i}\left|f_j\right|\right)^r w_k\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}$$

holds, it is sufficient to prove the validity of (1) for a non-negative sequence $f \in l_{p,v}$.

For all $i \geq 1$ we consider the following set:

$$T_i = \left\{ l \in Z : \sum_{k=1}^{i} \left(\sum_{j=k}^{i} f_j \right)^r w_k \geq 2^{rl} \right\},$$

where Z is the set of integers. Assume that $\theta_i = \max T_i$. Then for any $i \geq 1$:

$$2^{r\theta_i} \leq \sum_{k=1}^{i} \left(\sum_{j=k}^{i} f_j \right)^r w_k \leq 2^{r(\theta_i+1)}. \tag{6}$$

Let $m_1 = 1$ and $M_1 = \{ i \in N : \theta_i = \theta_{m_1} = \theta_1 \}$. We will define the value m_2 as $m_2 = \sup M_1 + 1$. Obviously $m_2 > m_1$. If the set M_1 is bounded from above, then $m_2 < \infty$ and $m_2 = \max M_1 + 1$. Let $1 = m_1 < m_2 < \ldots < m_s < \infty$ be inductively determined for $s \geq 1$. Then assume that $m_{s+1} = \sup M_s + 1$ to determine the value m_{s+1}, where $M_s = \{ i \in N : \theta_i = \theta_{m_s} \}$.

Let $N_0 = \{ s \in N : m_s < \infty \}$. For convenience, we introduce the notation $\theta_{m_s} = n_s$. Then from the definition of m_s and by (6) for $s \in N_0$, we have

$$2^{rn_s} \leq \sum_{k=1}^{i} \left(\sum_{j=k}^{i} f_j \right)^r w_k \leq 2^{r(n_s+1)}, \quad m_s \leq i \leq m_{s+1} - 1, \tag{7}$$

and

$$N = \bigcup_{s \in N_0} [m_s, m_{s+1}), \quad [m_s, m_{s+1}) \bigcap_{s \neq l} [m_l, m_{l+1}) = \varnothing.$$

Therefore, we can write left-hand side of inequality (1) as follows:

$$S := \sum_{i=1}^{\infty} u_i \left(\sum_{k=1}^{i} \left| \sum_{j=k}^{i} f_j \right|^r w_k \right)^{\frac{q}{r}} = \sum_{s \in N_0} \sum_{i=m_s}^{m_{s+1}-1} u_i \left(\sum_{k=1}^{i} \left| \sum_{j=k}^{i} f_j \right|^r w_k \right)^{\frac{q}{r}}. \tag{8}$$

If $m_s = \infty$, then we assume that $\sum_{i=m_s}^{m_{s+1}-1} = 0$. Taking into account (7) and (8), we estimate the value S:

$$S := \sum_{s \in N_0} \sum_{i=m_s}^{m_{s+1}-1} u_i \left(\sum_{k=1}^{i} \left| \sum_{j=k}^{i} f_j \right|^r w_k \right)^{\frac{q}{r}} \leq \sum_{s \in N_0} \sum_{i=m_s}^{m_{s+1}-1} u_i 2^{q(n_s+1)} = 2^{2q} \sum_{s \in N_0} 2^{q(n_s-1)} \sum_{i=m_s}^{m_{s+1}-1} u_i. \tag{9}$$

Next, we estimate the value S for cases 1) $r \geq 1$, 2) $0 < r < 1$ of parameter r separately:

1) Case $r \geq 1$. We estimate the value 2^{n_s-1} from above using inequality $n_{s-2} + 1 \leq n_s - 1$, which is derived from $n_{s-2} < n_{s-1} < n_s$, and estimation (7):

$$2^{n_s-1}=2^{n_s}-2^{n_s-1}\le 2^{n_s}-2^{n_{s-2}+1}\le\left(\sum_{k=1}^{m_s}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}-\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=k}^{m_{s-1}-1}f_j\right|^r w_k\right)^{\frac{1}{r}}.$$

Using Jensen's and Minkowski's inequalities for $r\ge 1$:

$$2^{n_s-1}\le\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k+\sum_{k=m_{s-1}}^{m_s}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}-\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=k}^{m_{s-1}-1}f_j\right|^r w_k\right)^{\frac{1}{r}}\le$$

$$\le\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=k}^{m_{s-1}-1}f_j+\sum_{j=m_{s-1}}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}+\left(\sum_{k=m_{s-1}}^{m_s}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}-\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=k}^{m_{s-1}-1}f_j\right|^r w_k\right)^{\frac{1}{r}}\le$$

$$\le\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=k}^{m_{s-1}-1}f_j\right|^r w_k\right)^{\frac{1}{r}}+\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=m_{s-1}}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}+\left(\sum_{k=m_{s-1}}^{m_s}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}-\left(\sum_{k=1}^{m_{s-1}-1}\left|\sum_{j=k}^{m_{s-1}-1}f_j\right|^r w_k\right)^{\frac{1}{r}}=$$

$$=\left(\sum_{k=1}^{m_{s-1}-1}w_k\right)^{\frac{1}{r}}\left(\sum_{j=m_{s-1}}^{m_s}f_j\right)+\left(\sum_{k=m_{s-1}}^{m_s}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}. \tag{10}$$

According to (9) and (10), we obtain the following inequality:

$$S\le 2^{2q}\sum_{s\in N_0}2^{q(n_s-1)}\sum_{i=m_s}^{m_{s+1}-1}u_i\le 2^{2q}\sum_{s\in N_0}\left[\left(\sum_{k=1}^{m_{s-1}-1}w_k\right)^{\frac{1}{r}}\left(\sum_{j=m_{s-1}}^{m_s}f_j\right)+\left(\sum_{k=m_{s-1}}^{m_s}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k\right)^{\frac{1}{r}}\right]^q\sum_{i=m_s}^{m_{s+1}-1}u_i\approx$$

$$\approx\sum_{s\in N_0}\left(\sum_{k=1}^{m_{s-1}-1}w_k\right)^{\frac{q}{r}}\left(\sum_{j=m_{s-1}}^{m_s}f_j\right)^q\sum_{i=m_s}^{m_{s+1}-1}u_i+\sum_{s\in N_0}\left(\sum_{k=m_{s-1}}^{m_s}\left|\sum_{j=k}^{m_s}f_j\right|^r w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{m_{s+1}-1}u_i=:S_1+S_2. \tag{11}$$

2) Case $0<r<1$. At first, we will estimate the value $2^{(n_s-1)r}$ from above using inequality $n_{s-2}+1\le n_s-1$ and estimation (7):

$$2^{(n_s-1)r}=\frac{2^{n_s r}-2^{(n_s-1)r}}{2^r-1}\le\frac{2^{n_s r}-2^{(n_{s-2}+1)r}}{2^r-1}\le\frac{1}{2^r-1}\left[\sum_{k=1}^{m_s}\left(\sum_{j=k}^{m_s}f_j\right)^r w_k-\sum_{k=1}^{m_{s-1}-1}\left(\sum_{j=k}^{m_{s-1}-1}f_j\right)^r w_k\right].$$

Using Jensen's inequality for $0<r<1$ we obtain

$$2^{(n_s-1)r}\le\frac{1}{2^r-1}\left[\sum_{k=1}^{m_{s-1}-1}\left(\sum_{j=k}^{m_{s-1}-1}f_j+\sum_{j=m_{s-1}}^{m_s}f_j\right)^r w_k+\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s}f_j\right)^r w_k-\sum_{k=1}^{m_{s-1}-1}\left(\sum_{j=k}^{m_{s-1}-1}f_j\right)^r w_k\right]\le$$

$$\leq \frac{1}{2^r-1}\left[\sum_{k=1}^{m_{s-1}-1}\left(\sum_{j=k}^{m_{s-1}-1} f_j\right)^r w_k+\sum_{k=1}^{m_{s-1}-1}\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^r w_k+\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s} f_j\right)^r w_k-\sum_{k=1}^{m_{s-1}-1}\left(\sum_{j=k}^{m_{s-1}-1} f_j\right)^r w_k\right]<<$$

$$<<\sum_{k=1}^{m_{s-1}-1}\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^r w_k+\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s} f_j\right)^r w_k=\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^r\sum_{k=1}^{m_{s-1}-1} w_k+\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s} f_j\right)^r w_k. \quad (12)$$

According to (9) and (12):

$$S<<\sum_{s\in N_0}2^{q(n_s-1)}\sum_{i=m_s}^{m_{s+1}-1}u_i\leq\sum_{s\in N_0}\left(\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^r\sum_{k=1}^{m_{s-1}-1} w_k+\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s} f_j\right)^r w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{m_{s+1}-1}u_i\approx$$

$$\approx\sum_{s\in N_0}\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^q\left(\sum_{k=1}^{m_{s-1}-1} w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{m_{s+1}-1}u_i+\sum_{s\in N_0}\left(\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s} f_j\right)^r w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{m_{s+1}-1}u_i:=S_1+S_2. \quad (13)$$

If we look at the estimates (11) and (13), in both cases of the parameter r, S is estimated with the same value from above, i.e.

$$S<<S_1+S_2. \quad (14)$$

We estimate the values S_1, S_2 for cases (i)-(v) separately.

(i) Let $0<p\leq 1$, $0<p\leq\min\{q,r\}<\infty$. We estimate the value S_1 from above:

$$S_1:=\sum_{s\in N_0}\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^q\left(\sum_{k=1}^{m_{s-1}-1} w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{m_{s+1}-1}u_i\leq\sum_{s\in N_0}\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j^{\frac{1}{p}} v_j^{-\frac{1}{p}}\right)^q\left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{\infty}u_i\leq$$

$$\leq\sum_{s\in N_0}\left(\sup_{j\in[m_{s-1},m_s]} v_j^{-\frac{1}{p}}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j^{\frac{1}{p}}\right)^q\left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{\infty}u_i\leq$$

$$\leq\sum_{s\in N_0}\left(U_{m_s}^-\right)^q\left(\sup_{j\in[m_{s-1},m_s]}\left(\sum_{k=1}^{j} w_k\right)^{\frac{1}{r}} v_j^{-\frac{1}{p}}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j^{\frac{1}{p}}\right)^q\leq$$

$$\leq\sum_{s\in N_0}\left(U_{m_s}^-\right)^q\left(\sup_{j\leq m_s}\left(\sum_{k=1}^{j} w_k\right)^{\frac{1}{r}} v_j^{-\frac{1}{p}}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j^{\frac{1}{p}}\right)^q\leq\sum_{s\in N_0}\left(A_{1,m_s}^-U_{m_s}^-\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j^{\frac{1}{p}}\right)^q\leq$$

$$\leq\sum_{s\in N_0}\sup_s\left(A_{1,m_s}^-U_{m_s}^-\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\leq G_1^q\sum_{s\in N_0}\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\leq G_1^q\left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \quad (15)$$

We get an upper estimate of the value S_2 using Hardy's inequality for $0<p\leq 1$, $0<p\leq r<\infty$:

$$S_2 := \sum_{s\in N_0}\left(\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s} f_j\right)^r w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{m_{s+1}-1}u_i \le \sum_{s\in N_0}\left(A^-_{m_{s-1},m_s}U^-_{m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le$$

$$\le \sum_{s\in N_0}\sup_s\left(A^-_{1,m_s}U^-_{m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_1^q\sum_{s\in N_0}\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_1^q\left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \qquad (16)$$

From inequalities (14), (15) and (16) we get the following estimate for case (i): $C_1 << G_1$.

(ii) Let $0<r<1<p\le q<\infty$. We estimate the value S_1 from above using Holder's inequality and equivalence (4):

$$S_1 := \sum_{s\in N_0}\left(\sum_{k=1}^{m_{s-1}-1} w_k\right)^{\frac{q}{r}}\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^q\sum_{i=m_s}^{m_{s+1}-1}u_i \le \sum_{s\in N_0}\left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}}\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j^{\frac{1}{p}} v_j^{-\frac{1}{p}}\right)^q\sum_{i=m_s}^{\infty}u_i \le$$

$$\le \sum_{s\in N_0}\left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}}\left(\sum_{j=m_{s-1}}^{m_s} v_j^{1-p'}\right)^{\frac{q}{p'}}\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\left(U^-_{m_s}\right)^q =$$

$$= \sum_{s\in N_0}\left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}}\left(\sum_{j=m_{s-1}}^{m_s} v_j^{1-p'}\right)^{\frac{q}{p'}\cdot\frac{p-r}{pr}\cdot\frac{pr}{p-r}}\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\left(U^-_{m_s}\right)^q \approx$$

$$\approx \sum_{s\in N_0}\left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}}\left(\sum_{j=m_{s-1}}^{m_s}\left(\sum_{i=j}^{m_s} v_i^{1-p'}\right)^{\frac{rp}{p'(p-r)}-1} v_j^{1-p'}\right)^{\frac{q(p-r)}{rp}}\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\left(U^-_{m_s}\right)^q \le$$

$$\le \sum_{s\in N_0}\left(\sum_{j=m_{s-1}}^{m_s}\left(\sum_{k=1}^{j} w_k\right)^{\frac{p}{p-r}}\left(\sum_{i=j}^{m_s} v_i^{1-p'}\right)^{\frac{p(r-1)}{p-r}} v_j^{1-p'}\right)^{\frac{q(p-r)}{rp}}\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\left(U^-_{m_s}\right)^q \le$$

$$\le \sum_{s\in N_0}\left(U^-_{m_s}\right)^q\left(\sum_{j=1}^{m_s}\left(\sum_{k=1}^{j} w_k\right)^{\frac{p}{p-r}}\left(\sum_{i=j}^{m_s} v_i^{1-p'}\right)^{\frac{p(r-1)}{p-r}} v_j^{1-p'}\right)^{\frac{q(p-r)}{rp}}\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le$$

$$\le \sum_{s\in N_0}\sup_s\left(B^-_{1,m_s}U^-_{m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_2^q\left(\sum_{s\in N_0}\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_2^q\left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \qquad (17)$$

By using Hardy's inequality for $0<r<1<p<\infty$ we obtain the following estimation of the value S_2:

$$S_2 := \sum_{s\in N_0}\left(\sum_{k=m_{s-1}}^{m_s}\left(\sum_{j=k}^{m_s} f_j\right)^r w_k\right)^{\frac{q}{r}}\sum_{i=m_s}^{m_{s+1}-1}u_i \le \sum_{s\in N_0}\left(B^-_{m_{s-1},m_s}U^-_{m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le$$

$$\le \sum_{s\in N_0} \sup_s \left(B^-_{1,m_s} U^-_{m_s}\right)^q \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_2^q \sum_{s\in N_0} \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_2^q \left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \qquad (18)$$

From inequalities (14), (17) and (18) in case (ii) we get the estimation $C_1 << G_2$.

(iii) Let $1 \le p \le \min\{q, r\} < \infty$. We will use Holder's inequality to estimate the value S_1:

$$S_1 \le \sum_{s\in N_0} \left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}} \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \left(\sum_{j=m_{s-1}}^{m_s} v_j^{1-p'}\right)^{\frac{q}{p'}} \sum_{i=m_s}^{\infty} u_i \le$$

$$\le \sum_{s\in N_0} \left(D^-_{1,m_s} U^-_{m_s}\right)^q \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} << G_3^q \left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \qquad (19)$$

We will estimate the value S_2 from above using Lemma 2.

$$S_2 := \sum_{s\in N_0} \left(D^-_{m_{s-1},m_s}\right)^q \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \sum_{i=m_s}^{m_{s+1}-1} u_i \le \sum_{s\in N_0} \left(D^-_{1,m_s} U^-_{m_s}\right)^q \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} <<$$

$$<< G_3^q \sum_{s\in N_0} \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_3^q \left(\sum_{s\in N_0} \sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \le G_3^q \left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \qquad (20)$$

From the estimates (14), (19) and (20), we get the estimate $C_1 << G_3$ for the inequality (1) in the case of (iii).

(iv) Let $0 < r < p$, $1 < p \le q < \infty$. To estimate the value of S_1, at first, we will use the Holder's inequality:

$$S_1 \le \sum_{s\in N_0} \left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}} \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \left(\sum_{j=m_{s-1}}^{m_s} v_j^{1-p'}\right)^{\frac{q}{p'}} \sum_{i=m_s}^{\infty} u_i <<$$

$$<< \sum_{s\in N_0} \left(\sum_{k=1}^{m_{s-1}} w_k \left(\sum_{j=1}^{k} w_j\right)^{\frac{r}{p-r}}\right)^{\frac{q(p-r)}{pr}} \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \left(\sum_{j=m_{s-1}}^{m_s} v_j^{1-p'}\right)^{\frac{q}{p'}} \sum_{i=m_s}^{\infty} u_i \le$$

$$\le \sum_{s\in N_0} \left(\sum_{k=1}^{m_{s-1}} w_k \left(\sum_{j=1}^{k} w_j\right)^{\frac{r}{p-r}} \left(\sum_{j=k}^{m_s} v_j^{1-p'}\right)^{\frac{r(p-1)}{p-r}}\right)^{\frac{q(p-r)}{pr}} \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} \sum_{i=m_s}^{\infty} u_i \le$$

$$\le \sum_{s\in N_0} \left(E^-_{1,m_s} U^-_{m_s}\right)^q \left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} << G_4^q \left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \qquad (21)$$

Now, we estimate the value S_2 from above:

$$S_2 := \sum_{s\in N_0}\left(E^-_{m_{s-1},m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\sum_{i=m_s}^{m_{s+1}-1}u_i \le \sum_{s\in N_0}\left(E^-_{1,m_s}U^-_{m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} <<$$

$$<< G_4^q\left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \tag{22}$$

From the estimates (14), (21) and (22), we get the estimate $C_1 << G_4$.

(v) Let $0 < r < p = 1 \le q < \infty$. Let us estimate the value S_1 from above.

$$S_1 \le \sum_{s\in N_0}\left(U^-_{m_s}\right)^q\left(\sum_{k=1}^{m_{s-1}} w_k\right)^{\frac{q}{r}}\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^q <<$$

$$<< \sum_{s\in N_0}\left(U^-_{m_s}\right)^q\left(\sum_{k=1}^{m_{s-1}} w_k\left(\sum_{j=1}^{k} w_j\right)^{\frac{r}{1-r}}\right)^{\frac{(1-r)q}{r}}\frac{\left(\min\limits_{m_{s-1}\le i\le m_s} v_i\right)^q}{\left(\min\limits_{m_{s-1}\le i\le m_s} v_i\right)^q}\left(\sum_{j=m_{s-1}}^{m_s} f_j\right)^q \le$$

$$\le \sum_{s\in N_0}\left(U^-_{m_s}\right)^q\left(\sum_{k=1}^{m_{s-1}} w_k\left(\sum_{j=1}^{k} w_j\right)^{\frac{r}{1-r}}\min_{k\le i\le m_s} v_i^{\frac{r}{r-1}}\right)^{\frac{(1-r)q}{r}}\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j\right)^q \le$$

$$\le \sum_{s\in N_0}\left(U^-_{m_s}\right)^q\left(\sum_{k=1}^{m_{s-1}} w_k\left(\sum_{j=1}^{k} w_j\right)^{\frac{r}{1-r}}\max_{k\le i\le m_s} v_i^{\frac{r}{1-r}}\right)^{\frac{(1-r)q}{r}}\left(\sum_{j=m_{s-1}}^{m_s} f_j v_j\right)^q \le$$

$$\le \sum_{s\in N_0}\left(F^-_{1,m_s}U^-_{m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} << G_5^q\left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}} \tag{23}$$

Now we estimate the value S_2:

$$S_2 := \sum_{s\in N_0}\left(F^-_{m_{s-1},m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}}\sum_{i=m_s}^{m_{s+1}-1}u_i \le \sum_{s\in N_0}\left(F^-_{1,m_s}U^-_{m_s}\right)^q\left(\sum_{j=m_{s-1}}^{m_s} f_j^p v_j\right)^{\frac{q}{p}} <<$$

$$<< G_5^q\left(\sum_{i=1}^{\infty} f_i^p v_i\right)^{\frac{q}{p}}. \tag{24}$$

From the estimates (14), (23), and (24), we get the estimate $C_1 << G_5$.

Hence in all five cases the inequality (1) and an estimation $C_1 << G_i$ $\left(i=\overline{1,5}\right)$ holds. From estimations (5) and $C_1 << G_i$ $\left(i=\overline{1,5}\right)$ we obtain that $C_1 \approx G_i$ $\left(i=\overline{1,5}\right)$.

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