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ON NON-INCREASING REARRANGEMENTS OF THE GENERALIZED FRACTIONAL MAXIMAL FUNCTION

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Abstract

The purpose of this article is to consider the symmetric rearrangement and non-increasing rearrangement of generalized fractional maximal functions. Concepts of rearrangement-invariant spaces and concepts of ideal spaces are considered. A generalized Lorentz-Morrey type space, in which the norm is determined by a symmetric rearrangement of functions, is considered. The equivalent norm for the function from the generalized Lorentz-Morrey space obtained. It is proved that in the definition of the norm in the generalized Lorentz-Morrey space, the internal norm from a symmetric rearrangement of a function over a ball centered at the point 0 can be replaced by the norm from a symmetric rearrangement of a function over a ball centered at an arbitrary point $x \in R^n$. A generalized fractional-maximal function a special case of which is a classical fractional-maximal function is considered. Estimates obtained for the non-increasing rearrangement of the generalized fractional maximal function. A pointwise estimate of the generalized fractional-maximal function by the generalized Riesz potential is obtained.

Keywords: non-increasing rearrangement, symmetric rearrangement of function, ideal spaces, generalized Lorentz-Morrey space, generalized fractional maximal function, generalized Riesz potential.

Аңдатпа

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ЖАЛПЫЛАНҒАН БӨЛШЕК-МАКСИМАЛДЫ ФУНКЦИЯНЫҢ ӨСПЕЙТІН АЛМАСТЫРУЛАРЫ ТУРАЛЫ

Бұл мақаланың мақсаты – жалпыланған бөлшек-максималды функциялардың симметриялық алмастыруын және өспейтін ауыстыруын қарастыру. Орын ауыстыру-инварианттық кеңістіктер және идеалды кеңістіктер ұғымдары қарастырылды. Лоренц-Морри типіндегі жалпыланған кеңістік қарастырылды, онда норма функциялардың симметриялы ауыстырылуы арқылы анықталады. Жалпыланған Лоренц-Морри кеңістігінен функцияның эквивалентті нормасы алынды. Жалпыланған Лоренц-Морри кеңістігіндегі норманы анықтауда центрі 0-де орналасқан шарды функцияның симметриялы ауыстырылуының ішкі нормасын функцияның центрі қандай да $x \in R^n$ нүктесіндегі шарға симметриялы ауыстыру нормасымен алмастыруға болатыны дәлелденді. Жалпыланған бөлшек-максималды функция қарастырылды, оның ерекше жағдайы классикалық бөлшек-максималды функция болып табылады. Жалпыланған бөлшек-максималды функцияның өспейтін ауыстыруы үшін бағалаулар алынды. Жалпыланған бөлшек-максималды функцияның жалпыланған Рисс потенциалымен нүктелік бағасы алынды.

Түйін сөздер: өспейтін орын ауыстыру, функцияның симметриялық ауыстырылуы, идеалды кеңістіктер, жалпыланған Лоренц-Морри кеңістігі, жалпыланған бөлшек-максималды функция, жалпыланған Рисс потенциалы.

Аннотация

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О НЕВОЗРАСТАЮЩИХ ПЕРЕСТАНОВКАХ ОБОБЩЕННОЙ ДРОБНО-МАКСИМАЛЬНОЙ ФУНКЦИИ

Целью статьи является рассмотрение симметрической перестановки и невозрастающей перестановки обобщенных дробно-максимальных функции. Рассматриваются концепции перестановочно-инвариантных пространств и концепция идеальных пространств. Рассматривается обобщенное пространство типа Лоренца-Морри, в котором норма определяется через симметрическую перестановку функций. Получена эквивалентная норма для функции из обобщенного пространства Лоренца-Морри. Доказывается, что в определении нормы в обобщенном пространстве Лоренца-Морри внутреннюю норму от симметрической перестановки функции по шару с центром в точке 0 можно заменить нормой от симметрической перестановки функции по шару с центром в произвольной точке $x \in R^n$. Рассмотрена обобщенная дробно-максимальная функция, частным случаем которой является классическая дробно-максимальная функция. Получены оценки для невозрастающей перестановки обобщенной дробно-максимальной функции. Получена поточечная оценка обобщенной дробно-максимальной функции через обобщенный потенциал Рисса.

Ключевые слова: невозрастающая перестановка, симметрическая перестановка функции, идеальные пространства, обобщенное пространство Лоренца-Морри, обобщенная дробно-максимальная функция, обобщенный потенциал Рисса.

1. Introduction

We consider the concepts introduced in the books of C.Bennett, R.Sharpley [1] and S.G.Crane, Yu.I. Petunin and E.M. Semenov [2]. In recent decades have been actively studied the theory of Morrey-type spaces and various integral operators in them. Detailed information can be found in the review articles by V.I. Burenkov [3-4]. In work V.I. Goldman and E.Bakhtygareeva [5] the generalized Lorentz-Morrey type spaces are considered. In this article, we show that in the definition of the norm in the generalized Lorentz-Morrey space given in [5], the internal norm of a symmetric rearrangement of a function over a ball centered at 0 can be replaced by the norm of a symmetric rearrangement of a function over a ball centered at an arbitrary point $x \in R^n$. Considered the generalized fractional-maximal function introduced in works [6], [7]. Other versions of the generalized fractional-maximal function are considered in the works of Hakim D.I., Nakai E., Sawano Y. [8] and Gogatishvili A., Pick L., Opic B. [9].

In this paper, we show that the generalized fractional-maximal function can be estimated from above in terms of the generalized Riesz potential. The generalized Riesz potential considered in [10], [11] and [12]. We have obtained estimates for a non-increasing rearrangement of a generalized fractional-maximal function. Similar estimates for the classical fractional-maximal function were previously obtained by Cianchi A., Kerman R., Opic B., Pick L. [13].

2. Preliminary information

We give a brief summary of Banach-functional spaces (briefly: BFS), rearrangement-invariant spaces (briefly: RIS).

Let S, Σ, μ be space with a measure. Here is Σ is σ -algebra of subsets of the set S, on which is determined a non-negative σ-finite, σ-additive measure μ. By $L_0 = L_0(S, \Sigma, \mu)$ denotes the set of μ-measurable real-valued functions $f: S \to R$, and by L_0^+ a subset of the set L_0 consisting of non-negative functions:

$$L_0^+ = \{f \in L_0 : f \geq 0\}.$$

In this work, we will use the concepts of a Banach-functional space (briefly: BFS), introduced by C.Bennett, R.Sharpley [1], as well as the concepts of an ideal space (briefly: IS) considered in the book by S.G.Crane, Yu.I.Petunin and E.M.Semenov [2].

Definition 1.1 [1]. A mapping $\rho: L_0^+ \to [0, \infty]$ is called a *functional norm (short: FN)*, if the next conditions are met for all $f, g, f_n \in L_0^+, n \in N$:

(P1) $\rho(f) = 0 \Rightarrow f = 0$, μ–almost everywhere (μ–a.e.);
$\rho(\alpha f) = \alpha \rho(f), \alpha \geq 0$; $\rho(f + g) \leq \rho(f) + \rho(g)$ (properties of the norm);

(P2) $f \leq g \ (\mu - a.e.) \Rightarrow \rho(f) \leq \rho(g)$ (monotony of the norm);

(P3) $f_n \uparrow f \Rightarrow \rho(f_n) \to \rho(f) \ (n \to \infty)$ (the Fatou property);

(P4) $0 < \mu(\sigma) < \infty \Rightarrow \int_\sigma f d\mu \leq C_\sigma \rho(f)$ (Local integrability);

(P5) $0 < \mu(\sigma) < \infty \Rightarrow \leq \rho(\chi_\sigma) < \infty$ (finiteness of the FN for characteristic functions χ_σ of sets of finite measure).

Here $f_n \uparrow f$ means that $f_n \leq f_{n+1}$, $\lim_{n\to\infty} f_n = f$ (μ–a.e.).

Definition 1.2 [2]. Let ρ there be a functional norm. The set of functions $X = X(\rho)$ from L_0, for which $\rho(|f|) < \infty$ is called a *Banach function space (briefly: BFS)*, generated by the FN ρ. For $f \in X$ we assume

$$\|f\|_X = \rho(|f|).$$

Definition 1.3 [2]. The space $X \in L_0(R^n)$ is called an *ideal space* if it satisfies the following conditions:

(B1) $\|f\| = 0 \Leftrightarrow f = 0$, μ–a.e., $\|\alpha f\| = \alpha\|f\|$, $\alpha \geq 0$;

$\exists C \in [1,\infty)$: $\|f + g\| \leq C(\|f\| + \|g\|)$; (1.1)

(B2) $0 \leq f \leq g\ (\mu - a.e.) \Rightarrow \|f\| \leq \|g\|$;

(B3) $0 \leq f_m (\mu - a.e.) \Rightarrow \|f_m\| \uparrow \|f\|$;

(B4) $\|f\| < \infty \Rightarrow |f| < \infty\ (\mu - a.e.)$. (1.1')

The space X is a normed space if $C = 1$ in triangle inequality (1.1), or a quasi-normed space if $C < 1$.

Recall that BFS satisfies properties (B1)-(B3) with $C = 1$ in (1.1'), property (B4) is replaced by more strict assumption:

(B4') $\Omega \in \mathrm{R}^n$, $|\Omega| \equiv \mu_n(\Omega) < \infty \Rightarrow \int_\Omega |f| d\mu_n \leq C_\Omega \|f\|$;

and the additional property holds:

(B5) $\Omega \in \mathrm{R}^n$, $|\Omega| \equiv \mu_n(\Omega) < \infty \Rightarrow \|\chi_\Omega\| < \infty$.

Therefore, the concept of an ideal space is broader than the concept of Banach function space.

Let $L_0 = L_0(R^n)$ be the set of all Lebesgue measurable functions $f : R^n \to C$ and μ_n be the Lebesgue measure on R^n. By L_0^+ we denote the subset of the set L_0 consisting of all non-negative functions:

$$L_0^+ = \{f \in L_0 : f \geq 0\}.$$

By $L_0^+(0,\infty;\downarrow)$ we denote the set of all non-increasing functions belonging to L_0^+. The non-increasing rearrangement f^* defined by the equality:

$$f^*(t) = \inf\{y \in [0,\infty) : \lambda_f(y) \leq t,\ t \in R_+ = (0,\infty)\}$$

where

$$\lambda_f(y) = \mu_n\{x \in R^n : |f(x)| > y,\ y \in [0,\infty)\}$$

is the Lebesgue distribution function.

It is known that $0 \leq f^* \downarrow$; $f^*(t+0) = f^*(t)$, $t \in R_+$; f^* is equally measurable with $|f|$, i.e.

$$\mu_1\{t \in R_+ : f^*(t) > y\} = \mu_n\{x \in R^n : |f(x)| > y\}.$$

Let $f^\#: \mathbb{R}^n \to \mathbb{R}^n$ denote the symmetric rearrangement of f, i.e. a radially symmetric non-negative non-increasing right-continuous function (as a function of $r = |x|$, $x \in \mathbb{R}^n$) which is equimeasurable with f. That is

$$f^\#(r) = f^*(v_n r^n);\ f^*(t) = f^\#\left(\left(\frac{t}{v_n}\right)^{\frac{1}{n}}\right),\ r,t \in \mathbb{R}_+,$$

here v_n is the volume of the n-dimensional unit ball.

The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau, \quad t \in R_+.$$

Note that f^{**} is an a non-decreasing function on R_+. Really, let $t_2 < t_1$, then

$$f^{**}(t_2)=\frac{1}{t_2}\int_0^{t_2} f^*(\tau)d\tau=\frac{1}{t_2}\int_0^{t_1} f^*(\tau)d\tau+\frac{1}{t_2}\int_{t_1}^{t_2} f^*(\tau)d\tau\le\frac{1}{t_2}\int_0^{t_1} f^*(\tau)d\tau+f^*(t_1)\frac{t_2-t_1}{t_2}.$$

Hence, we have

$$f^{**}(t_2)\le\frac{1}{t_2}\int_0^{t_1} f^*(\tau)d\tau+\frac{t_2-t_1}{t_2t_1}\int_0^{t_1} f^*(\tau)d\tau\le\left(\frac{1}{t_2}+\frac{t_2-t_1}{t_2t_1}\right)\int_{t_1}^{t_1} f^*(\tau)d\tau=\frac{1}{t_1}\int_0^{t_1} f^*(\tau)d\tau=f^{**}(t_1).$$

In addition, for $f\in\dot{L}_0$ we have: $\lambda_f(y)\to 0\ \ (y\to+\infty)\Leftrightarrow|f(x)|<\infty$, μ -a.e. on S.

Definition 1.4 [10]. Let ρ there be a functional norm. We say that ρ is *consistent* with the order relation $\prec$ if for $f,g\in L_0^+,\ f\prec g$ we have $\rho(f)\le\rho(g)$.

Note that by the property (P2), any FN is consistent with a pointwise estimate:

$$f\le g\ \ (\mu-a.e.)\Rightarrow\rho(f)\le\rho(g),$$

Definition 1.5 [10]. A FN ρ is *rearrangement-invariant* if

$$f^*\le g^*\Rightarrow\rho(f)\le\rho(g).$$

Banach function space (BFS) $X=X(\rho)$ generated by a rearrangement invariant functional norm ρ will be called a *rearrangement invariant space* (in short: RIS).

Definition 1.6 [5]. An ideal space $E=E(R^n)\subset L_0(R^n)$ is called a *generalized rearrangement invariant space* (briefly: GRIS) if the following additional propositions hold:

1. The (Quasi)norm $\|\cdot\|_{E(R^n)}$ depends only on the symmetric rearrangement of functions: namely

$$\|f\|_{E(R^n)}=\|f^{\#}\|_{E(R^n)},$$

2. E has additional properties:

(P6) $\Omega\in R^n$, $|\Omega|\equiv\mu_n(\Omega)<\infty\Rightarrow\varphi_E(|\Omega|)=\|\chi_\Omega\|_E<\infty$;

(P7) $\|\sigma_m\|_{E\to E}<\infty$, $m\in(1,\infty)$; $\sigma_m(f)(y)=f(m^{-1}y)$, $y\in R^n$.

Here φ_E is called the fundamental function for the GRIS $E=E(R^n)$, σ_m - the extension operator.

We consider a subspace in $L_0\equiv L_0(R^n)$ and in $\dot{L}_0\equiv\dot{L}_0(R^n)$, M_{EF} is a generalized version of a Lorentz-type space, respectively:

$$M_{EF}=\left\{f\in L_0(R^n):\|f\|_{M_{EF}}=\left\|\|f^{\#}\|_{E(B_t)}\right\|_F<\infty\right\}.$$

Lemma 1.1. Let $F=F(R_+)$ be an ideal space, and $E=E(R^n)$ be a generalized rearrangement invariant space. Then the following relation takes place:

$$\left\|\|f^{\#}\|_{E(B_t)}\right\|_F\approx\sup_{x\in R^n}\left\|\|f^{\#}\|_{E(B(x,t))}\right\|_F.$$

Proof. It is clear that

$$\left\|\|f^{\#}\|_{E(B_t)}\right\|_F\le\sup_{x\in R^n}\left\|\|f^{\#}\|_{E(B(x,t))}\right\|_F.$$

We prove the inverse inequality.

$$\left\|\|f^{\#}\|_{E(B_t)}\right\|_F\ge\sup_{x\in R^n}\left\|\|f^{\#}\|_{E(B(x,t))}\right\|_F.$$

$$\left\| f^{\#}(g)\chi_{B(x,t)}(y)\right\|_E \le \sup_{x\in R^n}\left\| f^{\#}(g)\chi_{B(0,t)}(y)\right\|_E,\ \ \forall x\in R^n .$$

$$\left\| f^{\#}(y)\chi_{B(x,t)}(y)\right\|_E = \left\| f^{\#}(y)\left[\chi_{B(0,t)\cap B(x,t)}(y)+\chi_{B(0,t)\cap B(x,t)}(y)\right]\right\|_E \le \left\| f^{\#}(y)\chi_{B(0,t)\cap B(x,t)}(y)\right\|_E +$$
$$+\left\| f^{\#}(y)\chi_{B(0,t)\cap B(x,t)}(y)\right\|_E \le \left\| f^{\#}(y)\chi_{B(0,t)}(y)\right\|_E + f^{\#}\left(v_n t^n\right)\left\|\chi_{B(x,t)}(y)\right\|_E \le \left\| f^{\#}(\cdot)\chi_{B(0,t)}(\cdot)\right\|_E + .$$
$$+ f^{\#}\left(v_n t^n\right)\left\|\chi_{B(x,t)}(y)\right\|_E.$$

Considering the assessment

$$f^{\#}\left(v_n t^n\right)\le f^{\#}(y),\ \ y\in B(0,t).$$

We have

$$f^{\#}\left(v_n t^n\right)\left\|\chi_{B(0,t)}(y)\right\|_E = \left\| f^{\#}\left(v_n t^n\right)\chi_{B(0,t)}(y)\right\|_E \le \left\| f^{\#}(y)\chi_{B(0,t)}(y)\right\|_E.$$

Therefore,

$$\left\| f^{\#}(y)\chi_{B(x,t)}(y)\right\|_E \le 2\left\| f^{\#}(y)\chi_{B(0,t)}(y)\right\|_E,\ \ \forall x\in R^n.$$

$$\sup_{x\in R^n}\left\|\left\| f^{\#}(y)\chi_{B(x,t)}(y)\right\|_E\right\|_F \le 2\left\|\left\| f^{\#}(y)\chi_{B(0,t)}(y)\right\|_E\right\|_F .$$

Lemma 1.1 is proved.

2. The generalized fractional-maximal function and estimate of its non-increasing rearrangement

Definition 2.1 [4]. Let $R\in(0,\infty]$, $R_+=(0,\infty)$. A function $\Phi:(0,R)\to R_+$ belongs to the class $B_n(R)$ if the following conditions hold:

(1) Φ is non-increasing and continuous on $(0,R)$;

(2) There exists a constant $C\in R_+$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \le C\Phi(r)r^n,\ \ r\in(0,R) \tag{2.1}$$

For example,

$$\Phi(\rho)=\rho^{\alpha-n}\in B_n(\infty)\ \ (0<\alpha<n);\quad \Phi(\rho)=ln\frac{eR}{\rho}\in B_n(R),\quad R\in R_+.$$

For $\Phi\in B_n(R)$ the following estimate also holds:

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \ge n^{-1}\Phi(r)r^n,\ \ r\in(0,R)$$

It is known that the following properties are performed for the $\Phi\in B_n(R)$ [4]:

$$r^{-n}\int_0^r \Phi(\rho)\rho^{n-1}d\rho \cong \Phi(r),\ \ r\in(0,R)$$
$$\Phi\in B_n(R)\Rightarrow \{0\le\Phi\downarrow : \Phi(r)r^n,\qquad r\in(0,R)\}.$$

For each $\alpha\in[1,\infty)$ there exists is a $\beta=\beta(\alpha,c,n)\in[1,\infty)$ (c is a constant from (2.1)) such that

$$\left\{\rho,r\in(0,R):\alpha^{-1}<\frac{\rho}{r}\le\alpha\right\}\Rightarrow \beta^{-1}<\frac{\Phi(\rho)}{\Phi(r)}\le\beta$$
$$(2.1)\Leftrightarrow \exists\gamma\in(0,n)\ \text{such that}\ \Phi(r)r^{\gamma}\ ess\uparrow\ on\ (0,R)$$

Now, we formulate the conditions on G.

Definition 2.2 [5]. Let $\Phi\in B_n(\infty)$. Say that $\psi\in S_\infty(\Phi)$ if

$$\psi^{\#}(\rho)\cong\Phi(\rho)\rho^n,\ \ \rho=|x|,\ \ x\in R_+,$$

say that $\psi \in S_\infty^0(\Phi)$ if

$$\psi(\rho) \cong \Phi(\rho)\rho^n,\ \rho = |x|,\ x \in R_+.$$

It is clear that $S_\infty^0(\Phi) \subset S_\infty(\Phi)$.

Let $\Phi \in B_n(\infty)$. The *generalized fractional-maximal function* $M_\Phi f$ is defined for the function $f \in L_1^{loc}(R^n)$ by the equality

$$M_\Phi f = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} f(y)dy,$$

where $B(x,r)$ is a ball with the center at the point x and radius r. That is, consider the operator $M_\Phi \in L_1^{loc}(R^n) \to \dot{L}_0(R^n)$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0,n)$ we obtain the classical fractional-maximal function $M_\alpha f$:

$$M_\alpha f = \sup_{t>0} \frac{1}{t^{n-\alpha}} \int\limits_{B(x,t)} |f(y)|dy.$$

We denote by $M_E^\Phi = M_E^\Phi(R^n)$ the set of the functions u, for which there is a function $f \in E(R^n)$ such that

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M_E^\Phi} = \inf\{\|f\|_E : f \in E(R^n),\ \ M_\Phi f = u\}.$$

Note that in the works [5], [10]-[12] the generalized Riesz potential was considered using the convolution operator:

$$A: E_1(R^n) \to \dot{L}_0(R^n),$$

$$Af(x) = (G * f)(x) = 2\pi^{-n/2} \int\limits_{R^n} G(x-y)f(y)dy,$$

where the kernel $G(x)$ satisfies the conditions:

$$G(x) \cong \Phi(|x|), \quad x \in R^n, \tag{2.2}$$

$$\Phi \in B_n(\infty); \quad \exists c \in R_+.$$

The kernel of the classical Riesz potential has the form

$$G(x) = |x|^{\alpha-n}, \quad \alpha \in (0,n). \tag{2.2}$$

In the following lemma, we prove that the generalized fractional-maximal function $M_\Phi f(x)$ is evaluated from above by the generalized Riesz potential.

Lemma 2.1. Let $\Phi \in B_n(\infty)$ and the kernel G defined by (2.2). Then the following inequality holds:

$$M_\Phi f(x) \le C(G * |f|)(x), \quad x \in R^n.$$

Proof of Lemma 2.1.

$$(G * f)(x) = 2\pi^{-n/2} \int\limits_{R^n} G(x-y)|f(y)|dy = \sup_{r>0} \int\limits_{B(x,r)} G(x-y)|f(y)|dy \gtrsim$$

$$\gtrsim \sup_{r>0} \operatorname*{ess\,inf}_{y\in B(x,r)} G(x-y) \int\limits_{B(x,r)} |f(y)|dy = \sup_{r>0} \operatorname*{ess\,inf}_{z\in B(0,r)} G(z) \int\limits_{B(x,r)} |f(y)|dy$$

$$= \sup_{r>0} \operatorname*{ess\,inf}_{t\in B(0,r)} \Phi(t) \int\limits_{B} |f(y)|dy = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy = M_\Phi f(x).$$

Lemma 2.1 is proved.

A function $f: R_+ \to R_+$ is called *quasi-decreasing (quasi-increasing)* if there exists a positive constant number $C > 1$ such that

$$f(t_2) < Cf(t_1) \ \text{ if } \ t_1 < t_2$$
$$(f(t_1) < Cf(t_2) \ \text{ if } \ t_1 > t_2).$$

Theorem 2.1. Let Φ be a positive, measurable, non-increasing function on $(0, \infty)$. Then the following inequality take place:

$$\sup_{t>0} \frac{1}{\Phi(t^{1/n})} (M_\Phi f)^*(t) \leq C \int_{R^n} |f(y)| dy. \qquad (2.3)$$

Proof of Theorem 2.1. Let $\lambda > 0$. Consider the set

$$E_\lambda = \{x \in R^n : (M_\Phi f)(x) > \lambda\}.$$

Then for each point $x \in E_\lambda$ we can find a ball $B(x) = B(x, t(x)) \subset B$, such that

$$\Phi(t(x)) \int_{B_x} |f(y)| dy > \lambda.$$

The family of balls $\{B_x\}_{x \in E}$ covers bounded sets E_λ. Then, by Vitali's lemma ([1]) about coverings, there is a sequence of pairwise non-intersecting balls $\{B_i\}_{i=1}^\infty$ and $\exists B_i = B(x_i, t_i)$: $B_i \cap B_j = \emptyset$, $E_\lambda \subset \bigcup_{i=1}^\infty 5\, B_i$ for which the inequality holds:

$$\Phi(|B_i|) \int_{B(x_i, t_i)} |f(y)| dy > \lambda.$$

Note that from the properties of the function $\Phi(t)$ follows that it is concave on $(0, \infty)$.

Therefore, for the function $\varphi(t)$:

$$\varphi(t) = \frac{1}{\Phi(t^{1/n})} \uparrow$$

holds the next inequality

$$\varphi\left(\sum_{i=1}^\infty t_i\right) \leq C \left(\sum_{i=1}^\infty \varphi(t_i)\right).$$

Hence

$$\lambda\varphi(|E_\lambda|) \leq \lambda\varphi\left(\sum_{i=1}^\infty |5B_i|\right) \leq \sum_{i=1}^\infty \lambda\varphi(5^n |B_i|) \leq C \cdot 5^n \sum_{i=1}^\infty \lambda\varphi(|B_i|)$$
$$\leq C \cdot 5^n \sum_{i=1}^\infty \left(\Phi(|B_i|) \int_{B_i} |f(y)| dy\right) \varphi(|B_i|) \leq C \cdot 5^n \sum_{i=1}^\infty \int_{B_i} |f(y)| dy \leq C \cdot 5^n \int_{R^n} |f(y)| dy.$$

Therefore, we got that for any $\lambda > 0$ there is an estimate:

$$\lambda\varphi(|E_\lambda|) \leq C \cdot 5^n \int_{R^n} |f(y)| dy.$$

Therefore

$$\sup_{\lambda>0} \lambda\varphi(|E_\lambda|) \leq C \cdot 5^n \int_{R^n} |f(y)| dy.$$

And this is equivalent to

$$\sup_{t>0} \varphi(t) (M_\Phi f)^*(t) \leq C \cdot 5^n \int_{R^n} |f(y)| dy.$$

Hence, inequality (2.3) take place. ***Theorem 2.1. is proved.***

Theorem 2.2. Let $\Phi \in B_n(\infty)$. Then the following estimate holds:

$$(M_\Phi f)^*(t) \le C\left(\Phi(\mathrm{t}^{1/n})\int_0^t f^*(u)du + \sup_{t\le s<\infty} s\Phi(s^{1/n})\,f^*(s)\right), \qquad t \in (0,\infty),$$

for every $f \in L_1^{loc}(R^n)$.

The proof of Theorem 2.2 is carried out using the above Theorem 2.1 and Theorem 2.1 from [6].

Conclusion

In this paper, we consider the generalized fractional maximal function and its non-increasing rearrangement and symmetric rearrangement. An estimate for a non-increasing rearrangement of generalized fractional maximal function is obtained in terms of a non-increasing rearrangement of that function. It is proved that generalized fractional maximal function is estimated from above in terms of the generalized Riesz potential. In addition, the norm of a function in spaces of the Lorentz-Morrey type is considered.

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