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APPLICATION OF THE FICTITIOUS REGION METHOD TO SOLVING A MODEL OCEANOLOGY PROBLEM

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Abstract

The study of the laws of fluid motion has always been an important aspect of the development of both technical and natural sciences. The solution to various problems arising in the analysis of fluid dynamics can be carried out both at the theoretical level and through carefully designed physical experiments. However, in many cases, creating models to study fluid phenomena is challenging, especially in laboratory or field studies. Physical experiments aimed at the detailed study of fluid motion often encounter technical difficulties and require significant resources and financial costs. In addition, the data obtained from such experiments are often limited in their applicability. This is why mathematical modeling plays a significant role in research in fluid dynamics. This makes it possible to more efficiently and cheaply study various aspects of fluid motion, and also provides the opportunity to apply the results obtained more widely. Modeling allows you to consider various factors affecting fluid movement and analyze their impact on the final result. Thus, mathematical modeling becomes an important tool for understanding and improving fluid movement concepts in various science and technology fields. This article discusses the fictitious domain method for a linear ocean flow problem. A generalized solution to the problem is given and its uniqueness is proved. The theorem of existence and convergence of solutions to approximate models obtained using the fictitious domain method are studied.

Keywords: fictitious domain method, hydrodynamics, oceanology, viscous fluid, irregular domain, stationary problems, finite difference method.

Аңдатпа

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ОКЕАНОЛОГИЯ МҮЛГІЛІК МӘСЕЛЕНІ ШЕШУ ҮШІН ЖАЛҒАН АЙМАҚТАР ӘДІСІН ҚОЛДАНУ

Сұйықтық қозғалысының заңдылықтарын зерттеу әрқашан техникалық және жаратылыстану ғылымдарының дамуының маңызды аспектісі болды. Сұйықтық динамикасын талдауда туындайтын әртүрлі мәселелерді шешу теориялық деңгейде де, мұқият құрастырылған физикалық тәжірибелер арқылы да жүзеге асырылуы мүмкін. Дегенмен, көптеген жағдайларда сұйықтық құбылыстарын зерттеу үшін модельдер жасау қиын, әсіресе зертханалық немесе далалық зерттеулерде. Сұйықтық қозғалысын егжей-тегжейлі зерттеуге бағытталған физикалық эксперименттер жиі техникалық қиындықтарға тап болады және айтарлықтай ресурстар мен қаржылық шығындарды талап етеді. Сонымен қатар, мұндай эксперименттерден алынған деректер көбінесе олардың қолданылуында шектеулі. Міне, сондықтан математикалық модельдеу сұйықтық динамикасы саласындағы зерттеулерде маңызды рөл атқарады. Бұл сұйықтық қозғалысының әртүрлі аспектілерін тиімдірек және арзанырақ зерттеуге мүмкіндік береді, сонымен қатар алынған нәтижелерді кеңінен қолдануға мүмкіндік береді. Модельдеу сұйықтықтың қозғалысына әсер ететін әртүрлі факторларды есепке алуға және олардың соңғы нәтижеге әсерін талдауға мүмкіндік береді. Осылайша, математикалық модельдеу ғылым мен техниканың әртүрлі салаларындағы сұйықтық қозғалысы туралы түсініктерді түсіну мен жетілдірудің маңызды құралына айналады. Бұл мақалада мұхит ағынының сызықтық мәселесі үшін жалған аймақтар әдісі талқыланады. Мәселенің жалпыланған шешімі келтіріліп, оның бірегейлігі дәлелденеді. Жалған домен әдісі арқылы алынған жуықталған модельдерге шешімдердің бар болуы және жинақтылығы теоремасы зерттеледі.

Түйін сөздер: жалған аймақтар әдісі, гидродинамика, океанология, тұтқыр сұйықтық, тұрақты емес домен, стационарлық есептер, шекті айырмашылықтар әдісі.

Аннотация

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ПРИМЕНЕНИЕ МЕТОДА ФИКТИВНЫХ ОБЛАСТЕЙ ДЛЯ РЕШЕНИЯ МОДЕЛЬНОЙ ЗАДАЧИ ОКЕАНОЛОГИИ

Изучение законов движения жидкостей всегда было важным аспектом в развитии как технических, так и естественных наук. Решение разнообразных задач, возникающих при анализе динамики жидкостей, может осуществляться как на теоретическом уровне, так и путем проведения тщательно разработанных физических экспериментов. Тем не менее, во многих случаях создание моделей для изучения явлений, связанных с движением жидкостей, представляет собой сложную задачу, особенно при проведении лабораторных или полевых исследований. Физические эксперименты, направленные на подробное изучение движения жидкости, часто сталкиваются с техническими сложностями, требуют значительных ресурсов и финансовых затрат. Кроме того, данные, полученные в результате таких опытов, зачастую ограничены в своей применимости. Именно поэтому математическое моделирование играет существенную роль в исследованиях в области гидродинамики. Это позволяет более эффективно и дешево исследовать различные аспекты движения жидкости, а также предоставляет возможность более широко применять полученные результаты. Моделирование позволяет учитывать разнообразные факторы, влияющие на движение жидкости, и анализировать их влияние на конечный результат. Таким образом, математическое моделирование становится важным инструментом для понимания и улучшения понятий о движении жидкостей в различных областях науки и техники. В данной статье рассматривается метод фиктивных областей для линейной задачи течения океана. Дается обобщенное решение задачи и доказывается его единственность. Исследованы теорема существования и сходимости решения приближенных моделей, полученных с помощью метода фиктивных областей.

Ключевые слова: метод фиктивных областей, гидродинамика, океанология, вязкая жидкость, нерегулярная область, стационарные задачи, метод конечных разностей.

Introduction

The study of processes occurring in the atmosphere and ocean is an important aspect of geophysics. When studying these phenomena, mathematical models based on systems of partial differential equations, mainly of the Navier-Stokes type, are actively used. Of particular interest are hydrodynamic models describing atmospheric processes, and important contributions to this area have been made by I.A. Kibel and his students [1].

Solving stationary problems of mathematical physics is an important part of computational mathematics. Some of them can be considered as limiting cases of non-stationary problems. When using asymptotic stationary methods, no attention is paid to the intermediate values of the solution, since they do not matter. Analytical methods leading to explicit solutions are rarely applicable, and approximate methods are most often used. This requires studying the correctness of boundary value problems for differential equations and their approximation, which includes the classical theory of differential equations and functional analysis [2].

Work by G.V. Demidov and G.I. Marchuk [3] was one of the first to study the correctness of mathematical models in meteorology and oceanology. Later this direction was developed in the works of Yu.Ya. Belov [4], B.A. Bubnov, A.V. Kazhikov, A.A. Kordzadze, V.I. Sukhonosov, Sh. Smagulov [5] and others.

In the work of V.P. Kochergin [7] studied a model of ocean dynamics in which the quasilinear terms $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ were absent in the first two equations of motion, and the seawater density diffusion equation was considered in full form.

One of the difficulties in numerically solving problems of mathematical physics is the arbitrariness of the domain boundary. To overcome this problem, the fictitious region method was proposed by V.K. Saulev. This idea was then developed in the works of V.Ya. Rivkinda, A.N. Konovalov, Sh. Smagulov and others.

Another feature of numerical methods for hydrodynamics problems is the non-evolutionary nature of the Navier-Stokes system of equations, which makes it difficult to use the effective method of fractional steps. In this regard, the idea of approximating the Navier-Stokes equations by equations of evolutionary type was put forward in the work of N.N. Vladimirov, B.G. Kuznetsov, N.N. Yanenko.

R. Temam proposed another method of ε-approximation, during which the behavior of the solution as ε→0 was studied and a difference scheme was developed that converged to the solution of the boundary value problem for the Navier-Stokes equations under certain conditions.

In the work of V.Ya. Rivkind presents various economic difference schemes such as fractional steps without introducing aditional terms $\frac{1}{2}\vec{v}^{\varepsilon}\mathrm{div}\vec{v}^{\varepsilon}$ into the equations of motion and using a formal modification of nonlinear terms.

In the work of Yu.Ya. Belov proved a theorem on the existence of a generalized solution of the linearized Navier-Stokes system with a small parameter and obtained estimates for the rate of convergence as ε→0.

In the listed studies, mainly weak generalized solutions were considered (most often from the class $\dot{W}_2^{1.1}(Q)$), but the work of P.E. Sobolevsky and V.V. Vasilyev presents for the first time a detailed study of a system with a small parameter, including the behavior of strong solutions.

Thus, one of the key directions in the development of mathematical modeling methods is associated with the study of approximate methods for solving complex multidimensional problems of mathematical physics. To effectively solve many applied problems associated with unstructured domains, the fictitious domain method is widely used, which is characterized by a high degree of automation in programming. The main concept of the fictitious domain method is to solve the problem not in the most complex initial domain D_0, but in a simpler domain D, where $D_0 \subset D$. In this paper, we present a stationary problem of studying the fictitious domain method for a linear equation in oceanology.

Formulation of the problem

The linear problem describing ocean currents is reduced to solving the following equations in the region $\Omega_0 = (0, H) \cdot D_0, \quad \Omega \subset R^2$:

$$\mu_0 \frac{\partial^2 u}{\partial z^2} + \mu\Delta u - \frac{\partial p}{\partial x} = f_1, \tag{1}$$

$$\mu_0 \frac{\partial^2 v}{\partial z^2} + \mu\Delta v - \frac{\partial p}{\partial y} = f_2, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0, \quad \frac{\partial p}{\partial z} = -\rho_0 g \tag{3}$$

with boundary conditions

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \omega = 0, \ \ with \ \ z = 0, \quad (x, y) \epsilon D_0 \tag{4}$$

$$\omega = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \ \ with \ \ z = H, \quad (x, y) \epsilon D_0 \tag{5}$$

$$u = v = 0 \ \text{ afloat } [0, H] \cdot \partial D_0, \tag{6}$$

where u,v is the velocity, ω is the vorticity in a limited simply connected region D_0 with the boundary ∂D_0.

We integrate the second equation (3) with respect to z:

$$\mathrm{p} = \mathrm{p}(x, y, 0) - \rho_0 \int_0^z g dz.$$

Denoting $\xi(x, y) = \mathrm{p}(x, y, 0)$ and integrating the first equation (3) $z \in [0, H]$ using conditions (4), (5), we write equations (1) – (6) in equivalent form:

$$\mu_0 \frac{\partial^2 u}{\partial z^2} + \mu\Delta u - \frac{\partial \xi}{\partial x} = F_1, \tag{7}$$

$$\mu_0 \frac{\partial^2 v}{\partial z^2} + \mu\Delta v - \frac{\partial \xi}{\partial y} = F_2, \tag{8}$$

$$\int_0^H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = 0, \quad \int_{D_0} \xi dxdy = 0, \frac{\partial \xi}{\partial z} = 0 \tag{9}$$

with boundary conditions

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \quad with \quad z = 0 \tag{10}$$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0, \quad with \quad z = \mathrm{H}$$

$u = v = 0$ in $[0, H] \cdot \partial D_0$,
were

$$F_1 = f_1 + \frac{\partial}{\partial x}\left(\int_0^z \rho_0 g dz\right), \qquad F_2 = f_2 + \frac{\partial}{\partial y}\left(\int_0^z \rho_0 g dz\right).$$

We will solve problem (7) – (10) using the fictitious domain method. Then we add the original area D_0 to some D_2, which can be, for example, a rectangle or a circle.

We introduce the notation: $\Omega_2 = (0, H) \cdot D_2$, $\Omega_1 = (0, H) \cdot D_1$, $D_1 = \frac{D_2}{D_0}$.

In the domain Ω_2 , we consider the auxiliary problem

in Ω_0

$$\mu_0 \frac{\partial^2 u^\varepsilon}{\partial z^2} + \mu \Delta u^\varepsilon - \frac{\partial \xi^\varepsilon}{\partial x} = F_1, \tag{11}$$

$$\mu_0 \frac{\partial^2 v^\varepsilon}{\partial z^2} + \mu \Delta v^\varepsilon - \frac{\partial \xi^\varepsilon}{\partial y} = F_2,$$

in Ω_1:

$$\mu_0 \frac{\partial^2 u^\varepsilon}{\partial z^2} + \frac{1}{\varepsilon} \mu \Delta u^\varepsilon - \frac{\partial \xi^\varepsilon}{\partial x} = F_1, \tag{12}$$

$$\mu_0 \frac{\partial^2 v^\varepsilon}{\partial z^2} + \frac{1}{\varepsilon} \mu \Delta v^\varepsilon - \frac{\partial \xi^\varepsilon}{\partial y} = F_2,$$

$$\int_0^H \left(\frac{\partial u^\varepsilon}{\partial x} + \frac{\partial v^\varepsilon}{\partial y}\right) dz = 0, \quad \int_{D_2} \xi^\varepsilon dxdy = 0, \quad \frac{\partial \xi^\varepsilon}{\partial z} = 0,$$

with boundary conditions

$$\frac{\partial u^\varepsilon}{\partial z} = \frac{\partial v^\varepsilon}{\partial z} = 0 \quad with \;\; z = H \tag{13}$$

$$\frac{\partial v^\varepsilon}{\partial z} = \frac{\partial u^\varepsilon}{\partial z} = 0 \quad with \;\; z = 0,$$

$$u^\varepsilon = v^\varepsilon = 0 \quad in \;\; [0, H] \cdot \partial D_2.$$

At the boundary ∂D_0 of the initial region we assume that the following matching conditions are satisfied:

$$\left\{\frac{1}{\varepsilon} \mu \left[\frac{\partial u^\varepsilon}{\partial x} \cos(\vec{n}, x) + \frac{\partial u^\varepsilon}{\partial y} \cos(\vec{n}, y)\right] - \xi^\varepsilon \cos(\vec{n}, x)\right\}\Big|_{\partial D_0^-} = \tag{14}$$

$$= \left\{\mu \left[\frac{\partial u^\varepsilon}{\partial x} \cos(\vec{n}, x) + \frac{\partial u^\varepsilon}{\partial y} \cos(\vec{n}, y)\right] - \xi^\varepsilon \cos(\vec{n}, x)\right\}\Big|_{\partial D_0^+};$$

$$\left\{\frac{1}{\varepsilon} \mu \left[\frac{\partial v^\varepsilon}{\partial x} \cos(\vec{n}, x) + \frac{\partial v^\varepsilon}{\partial y} \cos(\vec{n}, y)\right] - \xi^\varepsilon \cos(\vec{n}, x)\right\}\Big|_{\partial D_0^-} = \tag{15}$$

$$= \left\{\mu\left[\frac{\partial v^{\varepsilon}}{\partial x}\cos(\vec{n},x)+\frac{\partial v^{\varepsilon}}{\partial y}\cos(\vec{n},y)\right]-\xi^{\varepsilon}\cos(\vec{n},x)\right\}\Bigg|_{\partial D_0^+};$$

$$[\vec{u}^{\varepsilon}]|_{\partial D_0}=0. \qquad (16)$$

where $\vec{u}^{\varepsilon}=(u^{\varepsilon},v^{\varepsilon})$,, the signs “+”, “-” mean that the limiting values of the function on the curve ∂D_0 are taken from inside and outside the region D_0, respectively. In vector form, conditions (14), (15) can be written as follows:

$$\left[\frac{1}{\varepsilon}\mu\frac{\partial\vec{u}^{\varepsilon}}{\partial\vec{n}}-\xi^{\varepsilon}\vec{n}\right]\Bigg|_{\partial D_0^-}=\left[\mu\frac{\partial\vec{u}^{\varepsilon}}{\partial\vec{n}}-\xi^{\varepsilon}\vec{n}\right]\Bigg|_{\partial D_0^+}$$

We introduce the notation

$$\hat{C}^2(\Omega_2)=\left\{\left(u,v\in C^2(\Omega_2),(u,v)|_{\partial D_2}\right)=0,\quad z\in[0,H],\quad \frac{\partial u}{\partial z}=\frac{\partial v}{\partial z}\bigg|_{z=0}=0,\right.$$

$$\left.\frac{\partial u}{\partial z}\bigg|_{z=H}=\frac{\partial v}{\partial z}\bigg|_{z=H}=0,\ \int_0^H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz=0\right\};$$

$V_2(\Omega_2),\dot{V}_2^{\ell}(\Omega_2)$ – closure of $\hat{C}^2(\Omega_2)$ in the norm of the spaces $L_2(\Omega_2),W_2^{\ell}(\Omega_2)$, $\ell=-2;-1;0;1$.

Definition 1. A generalized solution to problem (11) – (16) is a function $\vec{u}^{\varepsilon}=(u^{\varepsilon},v^{\varepsilon})\in\dot{V}_2^1(\Omega_2)$ satisfying the following integral identity:

$$\int_0^H dz\int_{D_0}\left[\mu\frac{\partial\vec{u}^{\varepsilon}}{\partial z}\frac{\partial\vec{\varphi}}{\partial z}+\mu\nabla\vec{u}^{\varepsilon}\nabla\vec{\varphi}\right]dxdy+\int_0^H dz\int_{D_1}\left[\mu_0\frac{\partial\vec{u}^{\varepsilon}}{\partial z}\frac{\partial\vec{\varphi}}{\partial z}+\frac{\mu}{\varepsilon}\nabla\vec{u}^{\varepsilon}\nabla\vec{\varphi}\right]dxdy=\int_0^H dz\int_{D_0}\vec{F}\cdot\vec{\varphi}dxdy \quad (17)$$

$were\ \nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right),\vec{\varphi}\in\dot{V}_2^1(\Omega_2),\vec{F}=(F_1,F_2).$

Lemma 1. Let $\vec{F}(x,y,z)\in W_2^{-1}(\Omega_0)$. Then there is a unique generalized solution to problem (11) – (17) and for it the estimate holds:

$$\left\|\frac{\partial\vec{u}^{\varepsilon}}{\partial z}\right\|^2_{L_2(\Omega_2)}+\|\nabla\vec{u}^{\varepsilon}\|^2_{L_2(\Omega_0)}+\frac{1}{\varepsilon}\|\nabla\vec{u}^{\varepsilon}\|^2_{L_2(\Omega_1)}\le C\|\vec{F}\|^2_{W_2^{-1}(\Omega_0)} \qquad (18)$$

Proof of Lemma 1. We provide an estimate (18). To do this, multiply equations (11), (12) by $\vec{u}^{\varepsilon}$ and integrate by parts

$$\mu_0\left\|\frac{\partial\vec{u}^{\varepsilon}}{\partial z}\right\|^2_{L_2(\Omega_2)}+\mu_0\|\nabla\vec{u}^{\varepsilon}\|^2_{L_2(\Omega_0)}+\frac{1}{\varepsilon}\mu\|\nabla\vec{u}^{\varepsilon}\|^2_{L_2(\Omega_1)}+\int_0^H\int_{D_1}\xi\left(\frac{\partial u^{\varepsilon}}{\partial x}+\frac{\partial v^{\varepsilon}}{\partial y}\right)dxdydz=\int_{\Omega_0}\vec{F}\vec{u}^{\varepsilon}dx.$$

From here, using the Hölder and Young inequality, we obtain (18). The existence of a solution is proved by the Galerkin method. Uniqueness follows from (18).

Definition 2. A strong solution to problem (11) – (17) is the function $\vec{u}^{\varepsilon}\in\dot{V}_2^1(\Omega_2)\cap W_2^2(\Omega_i)$, $\xi^{\varepsilon}\in W_2^2(\Omega_i), i=0,1$, satisfying (11) – (17) almost everywhere.

Theorem 1. Let $\partial\Omega_0,\partial\Omega_2\in C^2$, $\vec{F}(x,y,z)\in L_2(\Omega_0)$. Then the weak solution to problem (11) – (17) is strong and the estimate holds

$$\left\|\frac{\partial^2\vec{u}^{\varepsilon}}{\partial z}\right\|_{L_2(\Omega_1)}+\|\vec{u}^{\varepsilon}\|_{W_2^2(\Omega_0)}+\frac{1}{\varepsilon}\|\vec{u}^{\varepsilon}\|_{L_2(0,H,W_2^2(D_1))}+\|\xi^{\varepsilon}\|_{W_2^1(\Omega_i)}\le C\|\vec{F}\|_{L_2(\Omega_0)},\qquad i=1,2 \qquad (19)$$

Proof of Theorem 1. We get (19).

Then we introduce the notation

$$\tilde{u}^{\varepsilon} = \int_0^H u^{\varepsilon} dz, \qquad \tilde{v}^{\varepsilon} = \int_0^H v^{\varepsilon} dz, \qquad \tilde{\xi}^{\varepsilon} = H\xi^{\varepsilon},$$

$$T_1 = \int_0^H F_1 dz + \mu_0 \frac{\partial u^{\varepsilon}}{\partial z}\Big|_{z=0}, \qquad T_2 = \int_0^H F_2 dz + \mu_0 \frac{\partial v^{\varepsilon}}{\partial z}\Big|_{z=0}.$$

Integrating (11), (12) over $z \in [0, H]$ we obtain

in D_0

$$\mu\Delta\tilde{u}^{\varepsilon} - \frac{\partial\tilde{\xi}^{\varepsilon}}{\partial x} = T_1, \quad (20)$$

$$\mu\Delta\tilde{v}^{\varepsilon} - \frac{\partial\tilde{\xi}^{\varepsilon}}{\partial y} = T_2,$$

in D_1

$$\frac{1}{\varepsilon}\mu\Delta\tilde{u}^{\varepsilon} - \frac{\partial}{\partial x}\tilde{\xi}^{\varepsilon} = 0, \quad (21)$$

$$\frac{1}{\varepsilon}\mu\Delta\tilde{v}^{\varepsilon} - \frac{\partial}{\partial y}\tilde{\xi}^{\varepsilon} = f,$$

$$\frac{\partial\tilde{u}^{\varepsilon}}{\partial x} + \frac{\partial\tilde{v}^{\varepsilon}}{\partial y} = 0.$$

In addition, the functions $\tilde{u}^{\varepsilon}, \tilde{v}^{\varepsilon}$ satisfy the boundary conditions

$$\tilde{u}^{\varepsilon}|_{\partial D_2} = \tilde{v}^{\varepsilon}|_{\partial D_2} = 0, \quad \tilde{u}^{\varepsilon}|_{\partial D_0} = \tilde{v}^{\varepsilon}|_{\partial D_0} = 0, \quad (22)$$

$$\left[\mu\frac{\partial\tilde{\vec{u}}^{\varepsilon}}{\partial\vec{n}} - \tilde{\xi}^{\varepsilon}\cdot\vec{n}\right]\Bigg|_{\partial D_0^+} = \left[\frac{\mu}{\varepsilon}\frac{\partial\tilde{\vec{u}}^{\varepsilon}}{\partial\vec{n}} - \tilde{\xi}^{\varepsilon}\cdot\vec{n}\right]\Bigg|_{\partial D_0^-}$$

Using known estimates of problem (20) – (22), we obtain

$$\|\tilde{\vec{u}}^{\varepsilon}\|_{W_2^2(D_0)} + \frac{1}{\varepsilon}\|\tilde{\vec{u}}^{\varepsilon}\|_{W_2^2(D_1)} + \|\tilde{\xi}^{\varepsilon}\|_{W_2^1(D_1)} + \|\tilde{\xi}^{\varepsilon}\|_{W_2^2(D_0)} \le C\|\vec{T}\|_{L_2(D_2)} \quad (23)$$

Then from equations (11), (12) we have the force

$$\mu_0\left\|\frac{\partial^2\vec{u}^{\varepsilon}}{\partial z^2}\right\|_{L_2(\Omega_2)} + \mu\|\Delta\vec{u}^{\varepsilon}\|_{L_2(\Omega_0)} + \frac{\mu}{\varepsilon}\|\Delta\vec{u}^{\varepsilon}\|_{L_2(\Omega_1)} \le C\left(\|\vec{F}\|_{L_2(\Omega_0)} + \|\nabla\xi^{\varepsilon}\|_{L_2(\Omega_2)}\right)$$

This inequality (23) guarantees estimate (19). The theorem has been proven.

Remark 1. Estimate (18) allows us to go to the limit as ε→0 in integral identity (17). From (18) it follows that from the sequence $\{\vec{u}^{\varepsilon}\}$ we can select a subsequence that weakly converges in $\dot{V}_2^1(\Omega_2)$.

Denotes its limit through $\vec{u}^0$. Passing to the limit in (17), it turns out that $\vec{u}^0$ corresponds to $\vec{u}$, i.e. It is a generalized solution to problems (7) – (10). From estimate (19) it follows that the strong solution of problem (11) – (17) as ε→0 approaches the strong solution of problem (7) – (10).

Then we estimate the rate of convergence of the solution to problem (11) – (17) as ε→0. Let $\vec{u}^{\varepsilon_1}$, $\vec{u}^{\varepsilon_2}$ be solutions to problems corresponding to parameters $\varepsilon_1, \varepsilon_2$. Takes the place of the next Theorem 2.

Theorem 2. Let the conditions of Lemma 1 be satisfied. Then the solution to problem (11) – (17) satisfies the estimate

$$\|\vec{u}^{\varepsilon_1} - \vec{u}^{\varepsilon_2}\|^2_{W_2^1(\Omega_0)} \le C_1(\varepsilon_1 + \varepsilon_2), \quad (24)$$

in which the constant C_1 depends on $\vec{F}(x, y, z)$ and does not depend on ε. Let us introduce the notation $\vec{u}^{\varepsilon_1} - \vec{u}^{\varepsilon_2} = \vec{\omega}$. By virtue of (11) – (17), the function $\vec{\omega}$ satisfies the relations:

$$\mu_0 \int_0^H dz \int_{D_2} \frac{\partial \vec{\omega}}{\partial z} \frac{\partial \vec{\varphi}}{\partial z} dxdy + \mu \int_0^H dz \int_{D_0} \nabla \vec{\omega} \cdot \nabla \vec{\varphi} dxdy + \frac{\mu}{\varepsilon_1} \int_0^H dz \int_{D_1} \nabla \vec{u}^{\varepsilon_1} \nabla \vec{\varphi} dxdy +$$

$$+ \frac{\mu}{\varepsilon_2} \int_0^H dz \int_{D_1} \nabla \vec{u}^{\varepsilon_2} \nabla \vec{\varphi} dxdy = 0, \qquad (25)$$

$$\int_0^H \left(\frac{\partial \omega_1}{\partial x} + \frac{\partial \omega_2}{\partial y} \right) dz = 0, \qquad \vec{\omega} = (\omega_1, \omega_2).$$

Let $\vec{\varphi} = \vec{\omega}$ in (25).

$$\mu_0 \left\| \frac{\partial \vec{\omega}}{\partial z} \right\|_{L_2(\Omega_2)}^2 + \mu \|\nabla \vec{\omega}\|_{L_2(\Omega_0)}^2 + \int_0^H dz \int_{D_1} \nabla \vec{\omega} \left(\frac{\mu}{\varepsilon_1} \nabla \vec{u}^{\varepsilon_1} + \frac{\mu}{\varepsilon_2} \nabla \vec{u}^{\varepsilon_2} \right) dxdy = 0.$$

Then we estimate the third term using Hölder's inequality and taking into account estimate (18)

$$\left| \int_0^H dz \int_{D_1} \left(\frac{\mu}{\varepsilon_1} \nabla \vec{u}^{\varepsilon_1} + \frac{\mu}{\varepsilon_2} \nabla \vec{u}^{\varepsilon_2} \right) \nabla \vec{\omega} dxdy \right| \leq \left[\frac{\mu}{\varepsilon_1} \|\nabla \vec{u}^{\varepsilon_1}\|_{L_2(\Omega_1)} + \frac{\mu}{\varepsilon_2} \|\nabla \vec{u}^{\varepsilon_2}\|_{L_2(\Omega_1)} \right] \cdot \|\nabla \vec{\omega}\|_{L_2(\Omega_1)} \leq$$

$$\leq C \|F\|_{W_2^{-1}(\Omega_0)} \cdot \|\nabla \vec{\omega}\|_{L_2(\Omega_1)} \leq C_2 (\varepsilon_1 + \varepsilon_2)$$

Theorem 2 is proven.
Consider the following version of the fictitious area method
in Ω_0

$$\mu_0 \frac{\partial^2 \vec{u}^\varepsilon}{\partial z^2} + \mu \Delta \vec{u}^\varepsilon - \nabla \xi^\varepsilon = \vec{F}, \qquad (26)$$

in Ω_1

$$\frac{1}{\varepsilon} \left(\mu_0 \frac{\partial^2 \vec{u}^\varepsilon}{\partial z^2} + \mu \Delta \vec{u}^\varepsilon \right) - \nabla \xi^\varepsilon = 0, \qquad (27)$$

with boundary conditions (13) – (16).

We define a generalized solution to this problem using the integral identity

$$\mu_0 \int_0^H dz \int_{D_0} \frac{\partial \vec{u}^\varepsilon}{\partial z} \frac{\partial \vec{\varphi}}{\partial z} dxdy + \mu \int_0^H dz \int_{D_0} \nabla \vec{u}^{\varepsilon_2} \nabla \vec{\varphi} dxdy + \qquad (28)$$

$$+ \frac{\mu_0}{\varepsilon} \int_0^H dz \int_{D_1} \frac{\partial \vec{u}^\varepsilon}{\partial z} \frac{\partial \vec{\varphi}}{\partial z} dxdy + \frac{\mu}{\varepsilon} \int_0^H dz \int_{D_1} \nabla \vec{u}^{\varepsilon_2} \nabla \vec{\varphi} dxdy = \int_0^H dz \int_{D_0} (\vec{F}, \vec{\varphi}) dxdy,$$

for all $\vec{\varphi}(x, y, z) \in \dot{V}_2^1(\Omega_2)$ similar to Definition 1.

Theorem 3. Let $\vec{F}(x, y, z) \in W_2^{-1}(\Omega_0)$. Then there is a unique generalized solution to problem (26), (27), (13) – (16) and the following estimates are valid:

$$\|\Delta \vec{u}^\varepsilon\|_{W_2^1(\Omega_0)} + \frac{1}{\varepsilon} \|\Delta \vec{u}^\varepsilon\|_{W_2^1(\Omega_1)} \leq \mathrm{C} \|\vec{F}\|_{W_2^{-1}(\Omega_0)} \qquad (29)$$

$$\|\vec{u}^\varepsilon - \vec{u}\|_{W_2^1(\Omega_0)} \leq C_3 \varepsilon \qquad (30)$$

Let $\vec{\varphi}$ equal to $\vec{u}^\varepsilon$ in identity (28). We get:

$$\mu_0 \|\Delta \vec{u}^\varepsilon\|_{W_2^1(\Omega_0)}^2 + \frac{\mu}{\varepsilon} \|\Delta \vec{u}^\varepsilon\|_{W_2^1(\Omega_1)}^2 \leq \int_0^{\mathrm{H}} dz \int_{D_0} \vec{F} \cdot \vec{u}^\varepsilon dxdy \leq \|\vec{F}\|_{W_2^{-1}(\Omega_0)} \|\vec{u}^\varepsilon\|_{W_2^1(\Omega_0)}$$

From here, using Young's inequality, it is easy to obtain (29). Estimate (30) is derived according to the scheme given for Theorem 2. The proof of Theorem 2 is complete.

Conclusion

The need and scientific significance of studying dynamic stability, predicting possible operating modes and their consequences are dictated by the need to improve environmental monitoring and the state of biological resources, predict the circulation of water masses and its variability in the medium term, and control the spread of pollution in the ocean. This approach is successfully used in oceanology, underwater acoustics and atmospheric physics and allows us to understand the nature of a number of hydrodynamic and acoustic phenomena, build models and obtain results that cannot be reproduced by other methods.

Thus, in the course of our research, the fictitious domain method for ocean dynamics models was considered and mathematically justified, in particular, the model problem was studied:

$$\mu_0 \frac{\partial^2 \vec{u}^\varepsilon}{\partial z^2} + \mu \Delta \vec{u}^\varepsilon - \nabla \xi^2 = \vec{F}, \qquad in\ \Omega_1$$

$$\frac{1}{\varepsilon}\left(\mu_0 \frac{\partial^2 \vec{u}^\varepsilon}{\partial z^2} + \mu \Delta \vec{u}^\varepsilon\right) - \nabla \xi^\varepsilon = 0, \qquad in\ \Omega_1$$

$$\int_0^H \mathrm{div}\vec{u}^\varepsilon dz = 0, \qquad \int_{D_2} \xi^\varepsilon dxdy = 0, \quad \frac{\partial \xi^\varepsilon}{\partial z} = 0$$

with boundary conditions

$$\frac{\partial \vec{u}^\varepsilon}{\partial z} = 0, \qquad with\ z = H, \qquad \frac{\partial \vec{u}^\varepsilon}{\partial z} = 0, \qquad with\ z = 0, \qquad \vec{u}^\varepsilon = 0\ with\ (x, y, z) \in [0, H] \cdot \partial D_2$$

At the boundary ∂D_0 of the source domain, the following matching conditions are assumed to be satisfied:

$$[\vec{u}^\varepsilon]|_{\partial D_0} = 0, \ \ z \in [0, H],$$

$$\left[\mu \frac{\partial \tilde{\vec{u}}^\varepsilon}{\partial \vec{n}} - \tilde{\xi}^\varepsilon \cdot \vec{n}\right]\Bigg|_{\partial D_0^+} = \left[\frac{\mu}{\varepsilon} \frac{\partial \tilde{\vec{u}}^\varepsilon}{\partial \vec{n}} - \tilde{\xi}^\varepsilon \cdot \vec{n}\right]\Bigg|_{\partial D_0^-}$$

the signs “+” and “–” mean that the limiting values of the function on the curve ∂D_0 are taken from inside and outside the region D_0, respectively.

A theorem for the existence of a weak solution to the presented problem was also proven and that, under certain conditions on $\partial D_0, \partial D_2$ and $\vec{F}$, it is strong and the following convergence estimate is established:

$$\|\vec{u}^{\varepsilon_1} - \vec{u}^{\varepsilon_2}\|_{W_2^1(\Omega_0)} \leq C(\varepsilon_1 + \varepsilon_2).$$

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