**МРНТИ 27.31.44**
**УДК 517.956**

**10.51889/2959-5894.2023.83.3.002**

# CONSTRUCTION OF A SOLUTION TO A BOUNDARY VALUE PROBLEM WITH N-D CONDITIONS FOR THE FOUR-DIMENSIONAL GELLERSTEDT EQUATION

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*Abstract*

This work is a continuation of the study of a number of problems on the question of unique solvability for a degenerate equation of elliptic type. The article considers the Gellerstedt equation generalized to four variables in an infinite domain. This equation has four hypersurfaces of degeneracy. Earlier, sixteen fundamental solutions were constructed for the equation under consideration. From the basic theory of differential equations, it is known that each fundamental solution can be used in solving its own boundary value problem. Thus, by means of the obtained fundamental solutions the N problem, the Dirichlet problem, and two boundary value problems with mixed conditions have already been solved. The goal of this work is to construct a unique solution to a boundary value problem with mixed conditions, where one condition is the Neumann condition and three Dirichlet conditions. This is the first time that a problem with this formulation has been solved. We obtain solution of the problem in explicit form, which contains of second-order Gaussian hypergeometric series. In solving the problem, methods of partial differential equations, the method of differentiation of hypergeometric functions, the Gauss-Ostrogradsky formula and the Boltz autotransformation formula are used. The obtained results have a theoretical nature and can be used for further development of the theory of partial differential equations and the theory of special functions.

**Keywords:** mixed boundary value problem; generalized Gellerstedt equation; Neumann and Dirichlet conditions; fundamental solutions; Gaussian hypergeometric series; Lauricella's function.

*Аннотация*

**ПОСТОРОЕНИЕ РЕШЕНИЯ КРАЕВОЙ ЗАДАЧИ С УСЛОВИЯМИ N-D ДЛЯ ЧЕТЫРЕХМЕРНОГО УРАВНЕНИЯ ГЕЛЛЕРСТЕДТА**

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Настоящая работа является продолжением исследования ряда задач на вопрос однозначной разрешимости для вырождающегося уравнения эллиптического типа. В статье рассматривается обобщенное до четырех переменных уравнение Геллерстедта в бесконечной области. Данное уравнение имеет четыре гиперповерхности вырождения. Ранее для рассматриваемого уравнения было построено шестнадцать фундаментальных решений. Из основной теории дифференциальных уравнений известно, что каждое фундаментальное решение может быть использовано в решении своей краевой задачи. Так, с помощью некоторых полученных фундаментальных решений уже были решены задач N, задача Дирихле, и две задачи со смешанными условиями. Целью настоящей работы является нахождение единственного решения краевой задачи со смешанными условиями, где одним условием взято условие Неймана и три условия Дирихле. Задача с такой постановкой решается впервые. Решение задачи получено в явном виде, содержит гипергеометрические ряды Гаусса второго порядка. При решении задачи используются методы дифференциальных уравнений в частных производных, метод дифференцирования гипергеометрических функций, формула Гаусса-Остроградского и формула автотрансформации Больца. Полученные результаты имеют теоретический характер и могут быть использованы для дальнейшего развития теории дифференциальных уравнений с частными производными и теории специальных функций.

**Ключевые слова:** смешанная краевая задача; обобщенное уравнение Геллерстедта; условия Неймана и Дирихле; фундаментальные решения; гипергеометрический ряд Гаусса; функция Лауричеллы.

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**ТӨРТ ӨЛШЕМДІ ГЕЛЛЕРСТЕДТ ТЕҢДЕУІ ҮШІН N-D ШАРТТАРЫ БАР ШЕТТІК ЕСЕПТІҢ ШЕШІМІН ҚҰРУ**

Бұл жұмыс эллиптикалық типті азғындалған теңдеу үшін бірегей шешілімділік мәселе бойынша бірқатар есептерді зерттеудің жалғасы болып табылады. Мақалада шексіз облыстағы төрт айнымалыға жалпыланған Геллерстедт теңдеуі қарастырылады. Бұл теңдеуде төрт азғындау гипербеті бар. Бұған дейін қарастырылып отырған теңдеу үшін он алты іргелі шешім құрастырылған. Дифференциалдық теңдеулердің негізгі теориясынан әрбір іргелі шешімді өзінің шекаралық есебін шешуде қолдануға болатыны белгілі. Осылайша, кейбір алынған іргелі шешімдердің көмегімен N есебі, Дирихле есебі және шарттары аралас екі есеп шешілді. Бұл жұмыстың мақсаты бір Нейман және үш Дирихле аралас шарттары қойылған есепті шешу болып табылады. Есептің шешімі айқын түрде алынған және екінші ретті Гаусс гипергеометриялық қатарынан құрастырылған. Есепті шешу үшін дербес туындылы дифференциалдық теңдеулер әдістері, гипергеометриялық функцияларды дифференциалдау әдісі, Гаусс-Остроградский формуласы және Больц автотрансформация формуласы қолданылады. Алынған нәтижелер теориялық сипатқа ие және оны дербес туындылы дифференциалдық теңдеулер теориясы мен арнайы функциялар теориясын одан әрі дамыту үшін пайдалануға болады.

**Түйін сөздер:** аралас шеттік есеп; Геллерстедт жалпыланған теңдеуі; Нейман және Дирихле шарттары; іргелі шешімдер; Гаусс гипергеометриялық қатары; Лауричелла функциясы.

## 1. Introduction

Over the last century partial differential equations have a wide range of applications in solving problems in various fields, such as mechanics, physics, aerodynamics, acoustics, astronomy, etc. [1-3] In solving many problems, the so-called special functions of mathematical physics are used, these can be the Bessel, Hermite functions or the Gauss's, Appell's or Lauricella's hypergeometric functions. Paper [4] describes the application of hypergeometric series of many variables to research developments in the aerospace systems field. Hypergeometric functions are widely used in constructing the theory of simple and double layer potentials [5]. A number of boundary value problems for elliptic equations with singular coefficients in two-dimensional and three-dimensional spaces were studied by means of the constructed fundamental solutions [6-8]. The general properties of degenerate systems of second-order hypergeometric equations of Horn, Whittaker, Bessel and Laguerre were studied. Where all constructed normal-regular solutions expressed through the Humbert's function [9]. Scientists from leading research centers in many countries are studying special functions and researching their areas of application. This area of research is quite relevant, as evidenced by statistical data from the Web of Science database.

In this paper we consider the degenerate Gellerstedt equation of elliptic type

$$y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0,\ m,n,k,l > 0,\ m,n,k,l \equiv const$$

in domain $R_+^4 = \{(x,y,z,t): x>0, y>0, z>0, t>0\}$. Previously, fundamental solutions were constructed for the equation under consideration and their singularities were studied [10]. The obtained fundamental solutions consist of four-dimensional hypergeometric Lauricella's functions $F_A^{(4)}$ [11]. To date, using the obtained fundamental solutions several boundary value problems in an infinite domain and one problem in a bounded domain have been solved [12-16].

## 2. Statement of the problem

Consider the elliptic equation

$$H(u) = y^m z^k t^l u_{xx} + x^n z^k t^l u_{yy} + x^n y^m t^l u_{zz} + x^n y^m z^k u_{tt} = 0,\ m,n,k,l > 0 \qquad (2.1)$$

in domain $D = \{(x,y,z,t): x>0, y>0, z>0, t>0\}$, bounded by the following hyperplanes:

$$S_1=\{(0,y,z,t):x=0,y>0,z>0,t>0\},\ \ S_2=\{(x,0,z,t):x>0,y=0,z>0,t>0\},$$

$$S_3=\{(x,y,0,t):x>0,y>0,z=0,t>0\},\ \ S_4=\{(x,y,z,0):x>0,y>0,z>0,t=0\}.$$

The variables are tied by the following relation:

$$R^2=\frac{4}{(n+2)^2}x^{n+2}+\frac{4}{(m+2)^2}y^{m+2}+\frac{4}{(k+2)^2}z^{k+2}+\frac{4}{(l+2)^2}t^{l+2}.$$

**Problem** $ND_3$**.** Find a regular solution $u(x,y,z,t)$ of the equation (2.1) from the class $C(\bar{D})\cap C^1(D\cup \overline{S_1})\cap C^2(D)$, D) satisfying the conditions:

$$\left.\frac{\partial}{\partial x}u(x,y,z,t)\right|_{x=0}=\nu_1(y,z,t),\ (y,z,t)\in S_1, \tag{2.2}$$

$$\left.u(x,y,z,t)\right|_{y=0}=\tau_2(x,z,t),\ (x,z,t)\in\overline{S_2}, \tag{2.3}$$

$$\left.u(x,y,z,t)\right|_{z=0}=\tau_3(x,y,t),\ (x,y,t)\in\overline{S_3}, \tag{2.4}$$

$$\left.u(x,y,z,t)\right|_{t=0}=\tau_4(x,y,z),\ (x,y,z)\in\overline{S_4}, \tag{2.5}$$

$$\lim_{R\to\infty}u(x,y,z,t)=0, \tag{2.6}$$

where $\nu_1(y,z,t),\tau_2(x,z,t),\tau_3(x,y,t),\tau_4(x,y,z)$ are given continuous functions, moreover the function $\nu_1(y,z,t)$ at the origin of coordinates can go to integrable order infinity. Also, for the large enough values R, the following inequalities hold:

$$|\nu_1(y,z,t)|\le\frac{c_1}{\left[1+\frac{4}{(m+2)^2}y^{m+2}+\frac{4}{(k+2)^2}z^{k+2}+\frac{4}{(l+2)^2}t^{l+2}\right]^{\frac{1-2\alpha+\varepsilon_1}{2}}}, \tag{2.7}$$

$$|\tau_2(x,z,t)|\le\frac{c_6}{\left[1+\frac{4}{(n+2)^2}x^{n+2}+\frac{4}{(k+2)^2}z^{k+2}+\frac{4}{(l+2)^2}t^{l+2}\right]^{\varepsilon_6}} \tag{2.8}$$

$$|\tau_3(x,y,t)|\le\frac{c_7}{\left[1+\frac{4}{(n+2)^2}x^{n+2}+\frac{4}{(m+2)^2}y^{m+2}+\frac{4}{(l+2)^2}t^{l+2}\right]^{\varepsilon_7}} \tag{2.9}$$

$$|\tau_4(x,y,z)|\le\frac{c_8}{\left[1+\frac{4}{(n+2)^2}x^{n+2}+\frac{4}{(m+2)^2}y^{m+2}+\frac{4}{(k+2)^2}z^{k+2}\right]^{\varepsilon_8}} \tag{2.10}$$

here $c_1,c_6,c_7,c_8$ and $\varepsilon_1,\varepsilon_6,\varepsilon_7,\varepsilon_8$ are small enough positive numbers.

***Theorem 1.*** The boundary value problem $ND_3$ has at most one solution.

***Existence of a solution to*** $ND_3$ ***the problem.***

The solution to the $ND_3$ problem has the form

$$
\begin{aligned}
u(x_0,y_0,z_0,t_0) = &-\int_0^\infty\int_0^\infty\int_0^\infty y^m z^k t^l \nu_1(y,z,t)\, g_{15}(0,y,z,t;x_0,y_0,z_0,t_0)\,dydzdt + \\
&+\int_0^\infty\int_0^\infty\int_0^\infty x^n z^k t^l \tau_2(x,z,t)\frac{\partial}{\partial y} g_{15}(x,y,z,t;x_0,y_0,z_0,t_0)\Big|_{y=0} dxdzdt + \\
&+\int_0^\infty\int_0^\infty\int_0^\infty x^n y^m t^l \tau_3(x,y,t)\frac{\partial}{\partial z} g_{15}(x,y,z,t;x_0,y_0,z_0,t_0)\Big|_{z=0} dxdydt + \\
&+\int_0^\infty\int_0^\infty\int_0^\infty x^n y^m z^k \tau_4(x,y,z)\frac{\partial}{\partial t} g_{15}(x,y,z,t;x_0,y_0,z_0,t_0)\Big|_{t=0} dxdydz,
\end{aligned}
\tag{2.11}
$$

where

$$
g_{15}(x,y,z,t;x_0,y_0,z_0,t_0) = k_{15}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}\left(\frac{4}{k+2}\right)^{\frac{4}{k+2}}\left(\frac{4}{l+2}\right)^{\frac{4}{l+2}}\left(r^2\right)^{-\alpha+\beta+\gamma+\delta-4} yzty_0z_0t_0\times
$$
$$
\times F_A^{(4)}\left(4+\alpha-\beta-\gamma-\delta;\alpha,1-\beta,1-\gamma,1-\delta;2\alpha,2-2\beta,2-2\gamma,2-2\delta;\xi,\eta,\zeta,\varsigma\right)
$$

is fundamental solution to the equation (2.1). Here function $F_A^{(4)}$ is the Lauricella's function

$$
F_A^{(4)}(a;b_1,b_2,b_3,b_4;c_1,c_2,c_3,c_4;x,y,z,t) = \sum_{m,n,p,q}^{\infty}\frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_m(c_2)_n(c_3)_p(c_4)_q\, m!n!p!q!}x^m y^n z^p t^q,
$$

$\left(|x|+|y|+|z|+|t|<1\right)$,

$$
\begin{aligned}
k_{15} = &\frac{1}{4\pi^2}\left(\frac{4}{n+2}\right)^{2\alpha}\left(\frac{4}{m+2}\right)^{2\beta}\left(\frac{4}{k+2}\right)^{2\gamma}\left(\frac{4}{l+2}\right)^{2\delta}\times \\
&\times\frac{\Gamma(4+\alpha-\beta-\gamma-\delta)\Gamma(\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(1-\delta)}{\Gamma(2\alpha)\Gamma(2-2\beta)\Gamma(2-2\gamma)\Gamma(2-2\delta)}.
\end{aligned}
\tag{2.12}
$$

***Proof.*** Since the function $g_{15}$ is a fundamental solution to equation (2.1), it is obvious that the solution to problem (2.11) satisfies equation (2.1).

Let us prove that function (2.11) satisfies conditions (2.2) – (2.5) of the problem $ND_3$.

We present solution (2.11) in the following form

$$
u(x_0,y_0,z_0,t_0) = I_1(x_0,y_0,z_0,t_0)+I_2(x_0,y_0,z_0,t_0)+I_3(x_0,y_0,z_0,t_0)+I_4(x_0,y_0,z_0,t_0), \tag{2.13}
$$

where

$$
I_1(x_0,y_0,z_0,t_0) = -k_{15}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}\left(\frac{4}{k+2}\right)^{\frac{4}{k+2}}\left(\frac{4}{l+2}\right)^{\frac{4}{l+2}} y_0z_0t_0\int_0^\infty\int_0^\infty\int_0^\infty y^{m+1}z^{k+1}t^{l+1}\nu_1(y,z,t)\times
$$
$$
\times\left(r^2\right)^{-\alpha+\beta+\gamma+\delta-4} F_A^{(3)}\left(4+\alpha-\beta-\gamma-\delta;1-\beta,1-\gamma,1-\delta;2-2\beta,2-2\gamma,2-2\delta;\eta,\zeta,\varsigma\right)\Big|_{x=0} dydzdt
$$

$$
I_2(x_0,y_0,z_0,t_0) = k_{15}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}\left(\frac{4}{k+2}\right)^{\frac{4}{k+2}}\left(\frac{4}{l+2}\right)^{\frac{4}{l+2}} y_0z_0t_0\int_0^\infty\int_0^\infty\int_0^\infty x^n z^{k+1}t^{l+1}\tau_2(x,z,t)\times
$$
$$
\times\left(r^2\right)^{-\alpha+\beta+\gamma+\delta-4} F_A^{(3)}\left(4+\alpha-\beta-\gamma-\delta;\alpha,1-\gamma,1-\delta;2\alpha,2-2\gamma,2-2\delta;\xi,\zeta,\varsigma\right)\Big|_{y=0} dxdzdt
$$

$$I_3\left(x_0,y_0,z_0,t_0\right)=k_{15}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}\left(\frac{4}{k+2}\right)^{\frac{4}{k+2}}\left(\frac{4}{l+2}\right)^{\frac{4}{l+2}}y_0z_0t_0\int_0^\infty\int_0^\infty\int_0^\infty x^n y^{m+1}t^{l+1}\tau_3\left(x,y,t\right)\times$$
$$\times\left(r^2\right)^{-\alpha+\beta+\gamma+\delta-4}F_A^{(3)}\left(4+\alpha-\beta-\gamma-\delta;\alpha,1-\beta,1-\delta;2\alpha,2-2\beta,2-2\delta;\xi,\eta,\varsigma\right)\Big|_{z=0}dxdydt$$

$$I_4\left(x_0,y_0,z_0,t_0\right)=k_{15}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}\left(\frac{4}{k+2}\right)^{\frac{4}{k+2}}\left(\frac{4}{l+2}\right)^{\frac{4}{l+2}}y_0z_0t_0\int_0^\infty\int_0^\infty\int_0^\infty x^n y^{m+1}z^{k+1}\tau_4\left(x,y,z\right)\times$$
$$\times\left(r^2\right)^{-\alpha+\beta+\gamma+\delta-4}F_A^{(3)}\left(4+\alpha-\beta-\gamma-\delta;\alpha,1-\beta,1-\gamma;2\alpha,2-2\beta,2-2\gamma;\xi,\eta,\zeta\right)\Big|_{t=0}dxdydz$$

Let us check condition (2.2), for this we calculate the derivative of function (2.13) at $x_0\to 0$

$$\frac{\partial}{\partial x_0}u\left(x_0,y_0,z_0,t_0\right)=\frac{\partial}{\partial x_0}I_1\left(x_0,y_0,z_0,t_0\right)+\frac{\partial}{\partial x_0}I_2\left(x_0,y_0,z_0,t_0\right)+$$
$$+\frac{\partial}{\partial x_0}I_3\left(x_0,y_0,z_0,t_0\right)+\frac{\partial}{\partial x_0}I_4\left(x_0,y_0,z_0,t_0\right).$$

Let's consider the first integral and use the decomposition formula [17 p. 118, (14)]

$$F_A^{(3)}\left(a,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z\right)=\sum_{n_1,n_2,n_3=0}^{\infty}\frac{\left(a\right)_{n_1+n_2+n_3}\left(b_1\right)_{n_1+n_2}\left(b_2\right)_{n_1+n_3}\left(b_3\right)_{n_2+n_3}}{\left(c_1\right)_{n_1+n_2}\left(c_2\right)_{n_1+n_3}\left(c_3\right)_{n_2+n_3}n_1!n_2!n_3!}\times$$
$$\times x^{n_1+n_2}y^{n_1+n_3}z^{n_2+n_3}F\left(a+n_1+n_2,b_1+n_1+n_2;c_1+n_1+n_2;x\right)$$
$$\times F\left(a+n_1+n_2+n_3,b_2+n_1+n_3;c_2+n_1+n_3;y\right)F\left(a+n_1+n_2+n_3,b_3+n_2+n_3;c_3+n_2+n_3;z\right).$$

Then we get:

$$\frac{\partial}{\partial x_0}I_1\left(x_0,y_0,z_0,t_0\right)=k_{15}\left(\frac{4}{m+2}\right)^{\frac{4}{m+2}}\left(\frac{4}{k+2}\right)^{\frac{4}{k+2}}\left(\frac{4}{l+2}\right)^{\frac{4}{l+2}}y_0z_0t_0\int_0^\infty\int_0^\infty\int_0^\infty y^{m+1}z^{k+1}t^{l+1}\nu_1\left(y,z,t\right)\times$$
$$\times\left(r^2\right)^{-\alpha+\beta+\gamma+\delta-4}\left(r^2\right)^{3-\beta-\gamma-\delta}\left(r_2^2\right)^{-1+\beta}\left(r_3^2\right)^{-1+\gamma}\left(r_4^2\right)^{-1+\delta}P\left(0,y,z,t;x_0,y_0,z_0,t_0\right)\Big|_{x=0}dydzdt,\qquad(2.14)$$

where

$$P\left(0,y,z,t;x_0,y_0,z_0,t_0\right)=\sum_{l_1,l_2,l_3=0}^{\infty}\frac{\left(4+\alpha-\beta-\gamma-\delta\right)_{l_1+l_2+l_3}\left(1-\beta\right)_{l_1+l_2}\left(1-\gamma\right)_{l_1+l_3}\left(1-\delta\right)_{l_2+l_3}}{\left(2-2\beta\right)_{l_1+l_2}\left(2-2\gamma\right)_{l_1+l_3}\left(2-2\delta\right)_{l_2+l_3}l_1!l_2!l_3!}$$
$$\times\left(\frac{r_2^2-r^2}{r_2^2}\right)^{l_1+l_2}\left(\frac{r_3^2-r^2}{r_3^2}\right)^{l_1+l_3}\left(\frac{r_4^2-r^2}{r_4^2}\right)^{l_2+l_3}$$
$$\times F\left(-2-\alpha-\beta+\gamma+\delta,1-\beta+l_1+l_2;2-2\beta+l_1+l_2;\frac{r_2^2-r^2}{r_2^2}\right)$$
$$\times F\left(-2-\alpha+\beta-\gamma+\delta-l_2,1-\gamma+l_1+l_3;2-2\gamma+l_1+l_3;\frac{r_3^2-r^2}{r_3^2}\right)$$
$$\times F\left(-2-\alpha+\beta+\gamma-\delta-l_1,1-\delta+l_2+l_3;2-2\delta+l_2+l_3;\frac{r_4^2-r^2}{r_4^2}\right).$$

We use the following change of variables:

$$\frac{2}{m+2}y^{\frac{m+2}{2}}=\frac{2}{m+2}y_0^{\frac{m+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_1,\ \frac{2}{k+2}z^{\frac{k+2}{2}}=\frac{2}{k+2}z_0^{\frac{k+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_2,$$

$$\frac{2}{l+2}t^{\frac{l+2}{2}}=\frac{2}{l+2}t_0^{\frac{l+2}{2}}+\frac{2}{n+2}x_0^{\frac{n+2}{2}}s_3,$$

then from (2.14) at $x_0\to 0$ we have

$$\frac{\partial}{\partial x_0}I_1(x_0,y_0,z_0,t_0)=k_{15}\left(\frac{8}{n+2}\right)^{\frac{2}{n+2}}\left(\frac{4}{m+2}\right)^{-2\beta}\left(\frac{4}{k+2}\right)^{-2\gamma}\left(\frac{4}{l+2}\right)^{-2\delta}\nu_1(y_0,z_0,t_0)$$
$$\frac{\Gamma(1+\alpha)\Gamma(2-2\beta)\Gamma(2-2\gamma)\Gamma(2-2\delta)}{\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(1-\delta)\Gamma(4+\alpha-\beta-\gamma-\delta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ds_1ds_2ds_3}{\left(1+s_1^2+s_2^2+s_3^2\right)^{\alpha+1}}. \tag{2.15}$$

Let us calculate the integral from (2.15) using the following formula [18, p. 637(4.638-3)] and [19]

$$\int_0^\infty\int_0^\infty\cdots\int_0^\infty\frac{x_1^{p_1-1}x_2^{p_2-1}\dots x_n^{p_n-1}}{\left[1+(r_1x_1)^{q_1}+(r_2x_2)^{q_2}+\dots+(r_nx_n)^{q_n}\right]^s}dx_1dx_2\dots dx_n=$$
$$=\frac{\Gamma\left(\frac{p_1}{q_1}\right)\Gamma\left(\frac{p_2}{q_2}\right)\dots\Gamma\left(\frac{p_n}{q_n}\right)}{q_1q_2\dots q_nr_1^{p_1q_1}r_2^{p_2q_2}\dots r_n^{p_nq_n}}\frac{\Gamma\left(s-\frac{p_1}{q_1}-\frac{p_2}{q_2}-\dots-\frac{p_n}{q_n}\right)}{\Gamma(s)},\ (p_i>0,q_i>0,r_i>0,s>0),$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{ds_1ds_2ds_3}{\left(1+s_1^2+s_2^2+s_3^2\right)^{\alpha+1}}=8\int_0^\infty\int_0^\infty\int_0^\infty\frac{ds_1ds_2ds_3}{\left(1+s_1^2+s_2^2+s_3^2\right)^{\alpha+1}}=\frac{\pi^2\Gamma(2\alpha)}{\Gamma(\alpha+1)(2\alpha-1)2^{2\alpha-2}\Gamma(\alpha)}$$

By virtue of formulas, from (2.15) we obtain

$$\lim_{x_0\to 0}\frac{\partial}{\partial x_0}I_1(x_0,y_0,z_0,t_0)=4\pi^2k_{15}\left(\frac{4}{n+2}\right)^{-2\alpha}\left(\frac{4}{m+2}\right)^{-2\beta}\left(\frac{4}{k+2}\right)^{-2\gamma}\left(\frac{4}{l+2}\right)^{-2\delta}\nu_1(y_0,z_0,t_0)\times$$
$$\times\frac{\Gamma(2\alpha)\Gamma(2-2\beta)\Gamma(2-2\gamma)\Gamma(2-2\delta)}{\Gamma(4+\alpha-\beta-\gamma-\delta)\Gamma(\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)\Gamma(1-\delta)} \tag{2.16}$$

Considering definition (2.12) for $k_{15}$ from (2.16) we obtain $\lim\limits_{x_0\to 0}\frac{\partial}{\partial x_0}I_1(x_0,y_0,z_0,t_0)=\nu_1(y_0,z_0,t_0)$. It is not difficult to show that

$$\lim_{x_0\to 0}\frac{\partial}{\partial x_0}I_2(x_0,y_0,z_0,t_0)=0,\ \lim_{x_0\to 0}\frac{\partial}{\partial x_0}I_3(x_0,y_0,z_0,t_0)=0,\ \lim_{x_0\to 0}\frac{\partial}{\partial x_0}I_4(x_0,y_0,z_0,t_0)=0.$$

Therefore, $\lim\limits_{x_0\to 0}\frac{\partial}{\partial x_0}u(x_0,y_0,z_0,t_0)=\nu_1(y_0,z_0,t_0)$, then function (2.11) satisfies the condition (2.2) of the problem $ND_3$. It is not difficult to prove that function (2.11) satisfies conditions (2.3) – (2.5) of the problem $ND_3$.

Let us show that if the given functions for sufficiently large values of the argument satisfy inequalities (2.7) – (2.10), then the solution (2.11) of the problem $ND_3$ also satisfies condition (2.6). Indeed, let inequalities (2.7) – (2.10) be valid, make the following change of variables:

$$\xi_1 = \frac{1}{R_0}\frac{2}{n+2}x^{\frac{n+2}{2}},\ \eta_1 = \frac{1}{R_0}\frac{2}{m+2}y^{\frac{m+2}{2}},\ \zeta_1 = \frac{1}{R_0}\frac{2}{k+2}z^{\frac{k+2}{2}},\ \varsigma_1 = \frac{1}{R_0}\frac{2}{l+2}t^{\frac{l+2}{2}},$$

$$\sigma_1 = \frac{1}{R_0}\frac{2}{n+2}x_0^{\frac{n+2}{2}},\ \sigma_2 = \frac{1}{R_0}\frac{2}{m+2}y_0^{\frac{m+2}{2}},\ \sigma_3 = \frac{1}{R_0}\frac{2}{k+2}z_0^{\frac{k+2}{2}},\ \sigma_4 = \frac{1}{R_0}\frac{2}{l+2}t_0^{\frac{l+2}{2}},$$

here $R_0{}^2 = \frac{4}{(n+2)^2}x_0^{n+2} + \frac{4}{(m+2)^2}y_0^{m+2} + \frac{4}{(k+2)^2}z_0^{k+2} + \frac{4}{(l+2)^2}t_0^{l+2}.$

Then at $R_0 \to \infty$ from (2.7) – (2.10) we obtain the following inequalities for

$$\left|I_1(x_0,y_0,z_0,t_0)\right| \le \frac{k_{15}c_1}{R_0^{\varepsilon_1}} 4^{\frac{2}{2+m}+\frac{2}{2+k}+\frac{2}{2+l}} \left(\frac{2}{m+2}\right)^{-2\beta}\left(\frac{2}{k+2}\right)^{-2\gamma}\left(\frac{2}{l+2}\right)^{-2\delta}$$
$$\times \sigma_2^{\frac{2}{m+2}}\sigma_3^{\frac{2}{k+2}}\sigma_4^{\frac{2}{l+2}} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\eta_1\zeta_1\varsigma_1 d\eta_1 d\zeta_1 d\varsigma_1}{\left(1+\eta_1^2+\zeta_1^2+\varsigma_1^2\right)^{4+\alpha-\beta-\gamma-\delta}\left(\eta_1^2+\zeta_1^2+\varsigma_1^2\right)^{\frac{1-2\alpha+\varepsilon_1}{2}}}, \tag{2.17}$$

$$\left|I_2(x_0,y_0,z_0,t_0)\right| \le \frac{k_{15}c_6}{R_0^{2\varepsilon_6}} 4^{\frac{2}{2+m}+\frac{2}{2+k}+\frac{2}{2+l}} \left(\frac{2}{n+2}\right)^{-2\alpha}\left(\frac{2}{m+2}\right)^{\frac{2}{m+2}}\left(\frac{2}{k+2}\right)^{-2\gamma}\left(\frac{2}{l+2}\right)^{-2\delta}$$
$$\times \sigma_2^{\frac{2}{m+2}}\sigma_3^{\frac{2}{k+2}}\sigma_4^{\frac{2}{l+2}} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1^{2\alpha}\zeta_1\varsigma_1 d\xi_1 d\zeta_1 d\varsigma_1}{\left(1+\xi_1^2+\eta_1^2+\zeta_1^2+\varsigma_1^2\right)^{4+\alpha-\beta-\gamma-\delta}\left(\xi_1^2+\zeta_1^2+\varsigma_1^2\right)^{\varepsilon_6}}, \tag{2.18}$$

$$\left|I_3(x_0,y_0,z_0,t_0)\right| \le \frac{k_{15}c_7}{R_0^{2\varepsilon_7}} 4^{\frac{2}{2+m}+\frac{2}{2+k}+\frac{2}{2+l}} \left(\frac{2}{n+2}\right)^{-2\alpha}\left(\frac{2}{m+2}\right)^{-2\beta}\left(\frac{2}{k+2}\right)^{\frac{2}{k+2}}\left(\frac{2}{l+2}\right)^{-2\delta}$$
$$\times \sigma_2^{\frac{2}{m+2}}\sigma_3^{\frac{2}{k+2}}\sigma_4^{\frac{2}{l+2}} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1^{2\alpha}\eta_1\varsigma_1 d\xi_1 d\eta_1 d\varsigma_1}{\left(1+\xi_1^2+\eta_1^2+\zeta_1^2+\varsigma_1^2\right)^{4+\alpha-\beta-\gamma-\delta}\left(\xi_1^2+\eta_1^2+\varsigma_1^2\right)^{\varepsilon_7}}, \tag{2.19}$$

$$\left|I_4(x_0,y_0,z_0,t_0)\right| \le \frac{k_{15}c_8}{R_0^{2\varepsilon_8}} 4^{\frac{2}{2+m}+\frac{2}{2+k}+\frac{2}{2+l}} \left(\frac{2}{n+2}\right)^{-2\alpha}\left(\frac{2}{m+2}\right)^{-2\beta}\left(\frac{2}{k+2}\right)^{-2\gamma}\left(\frac{2}{l+2}\right)^{\frac{2}{l+2}}$$
$$\times \sigma_2^{\frac{2}{m+2}}\sigma_3^{\frac{2}{k+2}}\sigma_4^{\frac{2}{l+2}} \int_0^\infty\int_0^\infty\int_0^\infty \frac{\xi_1^{2\alpha}\eta_1\zeta_1 d\xi_1 d\eta_1 d\zeta_1}{\left(1+\xi_1^2+\eta_1^2+\zeta_1^2+\varsigma_1^2\right)^{4+\alpha-\beta-\gamma-\delta}\left(\xi_1^2+\eta_1^2+\zeta_1^2\right)^{\varepsilon_8}}. \tag{2.20}$$

Let us show that the integrals included in inequalities (2.17) - (2.20) are bounded.
Inequality (2.17) satisfies the identity

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{xyzdxdydz}{\left(1+x^2+y^2+z^2\right)^{4+a-b-c-d}\left(x^2+y^2+z^2\right)^{\frac{1-2\alpha+\varepsilon}{2}}} = \frac{1}{16}\frac{\Gamma\left(\frac{5-\varepsilon}{2}+a\right)\Gamma\left(\frac{3-\varepsilon}{2}-b-c-d\right)}{\Gamma(4+a-b-c-d)}, \tag{2.21}$$

$$где\ \ 2b+2c+2d-3<\varepsilon<2a+5.$$

For inequalities (2.18)-(2.20) the identity holds

$$\int_0^\infty\int_0^\infty\int_0^\infty \frac{x^{2a}yzdxdydz}{\left(1+x^2+y^2+z^2\right)^{4+a-b-c-d}\left(x^2+y^2+z^2\right)^{\varepsilon}}=\frac{1}{8}\frac{\Gamma\left(a+\frac{5-\varepsilon}{2}\right)\Gamma\left(2-b-c-d+\frac{\varepsilon}{2}\right)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(a+\frac{3}{2}\right)}{\Gamma\left(4+a-b-c-d\right)}, \quad (2.22)$$

$$где\ \ 2b+2c+2d-3<\varepsilon<2a+5.$$

Thus, from inequalities (2.17) – (2.20), due to the value of integrals (2.21) and (2.22), the following estimates hold:

$$\begin{aligned}&\lim_{R\to 0}\left|I_1\left(x_0,y_0,z_0,t_0\right)\right|\le\frac{k_{15}\overline{c_1}}{R_0^{\varepsilon_1}},\ \lim_{R\to 0}\left|I_2\left(x_0,y_0,z_0,t_0\right)\right|\le\frac{k_{15}\overline{c_6}}{R_0^{2\varepsilon_6}},\\&\lim_{R\to 0}\left|I_3\left(x_0,y_0,z_0,t_0\right)\right|\le\frac{\overline{c_7}}{R_0^{2\varepsilon_7}},\ \lim_{R\to 0}\left|I_4\left(x_0,y_0,z_0,t_0\right)\right|\le\frac{k_{15}\overline{c_8}}{R_0^{2\varepsilon_8}},\end{aligned} \quad (2.23)$$

where $\overline{c_5}, \overline{c_6}, \overline{c_7}, \overline{c_8}$ are constans.

Inequalities (2.23) show that solution (2.11) at $R_0\to\infty$ goes to zero. Thus, condition (2.6) of the problem $ND_3$ is satisfied. Consequently, solution (2.11) of the problem satisfies all the $ND_3$ problem conditions.

***Theorem 2.*** Let conditions (2.7) – (2.10) be satisfied, then a regular solution to problem $ND_3$ (2.1), (2.2) – (2.6) exists and is expressed by formula (2.11).

**Methodology**

To construct a solution to the boundary value problem under consideration, classical methods of partial differential equations and mathematical physics are used. Methods proposed by the authors of the famous monograph by Appell, Kampe de Feriet, were also used, applying the properties of the Gaussian hypergeometric function of one variable and Lauricella's function of four variables. By means of decomposition formulas, functions of many variables are reduced to the product of Gaussian functions of one variable. Integration results are written using Euler's Gamma function. The formula for differentiation of special functions, the properties of the Lauricella's function $F_D^{(4)}$ and the Pochhamer function is traditionally used.

**Conclusions**

A mixed boundary value problem $ND_3$ with one Neumann's condition and three Dirichlet's conditions is posed for an elliptic type equation in four-dimensional space. Two theorems are formulated. A theorem for the existence of a solution to the problem has been proven. The solution is obtained explicitly, expressed by the Lauricella's hypergeometric function $F_A^{(4)}$. During the proof, the properties of hypergeometric series, decomposition formulas and Boltz's formula, properties of Euler's Gamma function are used. In the future, it is possible to construct solutions to a number of mixed boundary value problems for the equation under consideration using the proposed method.

**Acknowledgments**

*This research was funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP14972818).*

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