IRSTI 27.31.44

10.51889/2959-5894.2023.84.4.002

O.S. Akhmetova[1], S.A. Issayev[2], A.N. Kaziyeva*

[1]Almaty brunch of Saint-Peterburg University of the Humanities and Social Sciences, Almaty, Kazakhstan
[2]Kazakh National Women's Teacher Training University, Almaty, Kazakhstan
3Abai Kazakh National Pedagogical University, Almaty, Kazakhstan
[*]e-mail: ah_oksa@mail.ru

SOLUTION OF THE NONLINEAR STATIONARY PROBLEM OF THE BAROCLINIC OCEAN BY THE FICTITIOUS DOMAIN METHOD

Abstract

Modeling is a key tool for understanding physical processes, analyzing the global spatial and temporal structure of the ocean, its interaction with the atmosphere, and regional variability in marine and ocean systems. Models also play an important role in processing and assimilating data from field observations. The development of mathematical modeling of ocean dynamics, which has more than a century of experience, has led to a significant increase in understanding of the physical processes occurring in the marine environment, as well as improved methods and models for their analysis. In addition, in recent years there has been increasing interest in studying the patterns of baroclinic fields, disturbances, and anomalies in the ocean, including the analysis of observational data, theoretical studies of the propagation of disturbances in a simplified oceanic environment, and numerical modeling. The basic principles of the theory of the baroclinic layer in the ocean can be derived from a complete set of primitive equations, including horizontal projections of the momentum balance equations, the hydrostatic equation, the mass conservation equation, the heat and salt diffusion equations, and the equation of state. This article discusses the fictitious domain method for the nonlinear stationary problem of the Baroclinic Ocean. A generalized solution to the problem is given and its uniqueness is proved. The theorem of existence and convergence of solutions to approximate models obtained using the fictitious domain method are studied.

Keywords: fictitious domain method, hydrodynamics, oceanology, viscous fluid, irregular domain, stationary problems, Baroclinic Ocean.

О.С. Ахметова[1], С.А. Исаев[2], А.Н. Казиева[3]
[1]Алматинский филиал Санкт-Петербургского Гуманитарного университета профсоюзов, Алматы, Казахстан
[2]Казахский национальный женский педагогический университет, Алматы, Казахстан
[3]Казахский национальный педагогический университет имени Абая, Алматы, Казахстан

РЕШЕНИЕ НЕЛИНЕЙНОЙ СТАЦИОНАРНОЙ ЗАДАЧИ БАРОКЛИННОГО ОКЕАНА МЕТОДОМ ФИКТИВНЫХ ОБЛАСТЕЙ

Аннотация

Моделирование является ключевым инструментом для осмысления физических процессов, анализа всемирной пространственно-временной структуры океана, его взаимодействия с атмосферой и региональной изменчивости морских и океанических систем. Модели также играют важную роль в обработке и ассимиляции данных из натурных наблюдений. Развитие математического моделирования динамики океана, имеющего более чем столетний опыт, привело к значительному углублению понимания физических процессов, происходящих в морской среде, а также к улучшению методов и моделей их анализа. Помимо этого, в последние годы усиливается интерес к исследованию закономерностей бароклинных полей, возмущений и аномалий в океане, включая анализ данных наблюдений, теоретическое исследование распространения возмущений в упрощенной океанической среде и численное моделирование. Основные принципы теории бароклинного слоя в океане могут быть выведены из полного набора примитивных уравнений, включающих горизонтальные проекции уравнений баланса количества движения, уравнение гидростатики, уравнение сохранения массы, уравнения диффузии тепла и соли, а также уравнение состояния. В данной статье рассматривается метод фиктивных областей для нелинейной стационарной задачи бароклинного океана. Дается обобщенное решение задачи и доказывается его единственность. Исследованы теорема существования и сходимости решения приближенных моделей, полученных с помощью метода фиктивных областей.

Ключевые слова: метод фиктивных областей, гидродинамика, океанология, вязкая жидкость, нерегулярная область, стационарные задачи, бароклинный океан.

О.С. Ахметова[1], С.А. Исаев[2], А.Н. Казиева[3]
[1]Санкт-Петербург Гуманитарлық кәсіподақтар университеті Алматы филиалы Алматы, Қазақстан
[2]Қазақ ұлттық қыздар педагогикалық университеті, Алматы, Қазақстан
[3]Абай атындағы Қазақ ұлттық педагогикалық университеті, Алматы, Қазақстан

БАРОКЛИНДІ МҰХИТТЫҢ СЫЗЫҚТЫ ЕМЕС СТАЦИОНАРЛЫҚ ЕСЕБІН ЖАЛҒАН АЙМАҚТЫҚ ӘДІСІМЕН ШЕШУ

Аңдатпа

Модельдеу физикалық процестерді түсінудің, мұхиттың ғаламдық кеңістіктік және уақыттық құрылымын, оның атмосферамен әрекеттесуін және теңіз және мұхит жүйелеріндегі аймақтық өзгермелілігін талдаудың негізгі құралы болып табылады. Модельдер далалық бақылаулардан алынған деректерді өңдеу мен ассимиляциялауда да маңызды рөл атқарады. Ғасырдан астам тәжірибесі бар мұхит динамикасын математикалық модельдеудің дамуы теңіз ортасында болып жатқан физикалық процестерді түсінудің айтарлықтай артуына, сондай-ақ оларды талдаудың әдістері мен үлгілерінің жетілдірілуіне әкелді. Сонымен қатар, соңғы жылдары мұхиттағы бароклинді өрістердің заңдылықтарын, бұзылулар мен аномалияларды зерттеуге, соның ішінде бақылау деректерін талдауға, жеңілдетілген мұхиттық ортада бұзылулардың таралуын теориялық зерттеуге және сандық модельдеуге қызығушылық артуда. Мұхиттағы бароклинді қабат теориясының негізгі принциптері импульс тепе-теңдік теңдеулерінің горизонталь проекцияларын, гидростатикалық теңдеулерді, массаның сақталу теңдеулерін, жылу мен тұздардың диффузия теңдеулерін қоса алғанда, қарабайыр теңдеулердің толық жиынтығынан алынуы мүмкін. күй теңдеуі. Бұл мақалада бароклинді мұхиттың сызықты емес стационарлық мәселесі үшін жалған домен әдісі талқыланады. Мәселенің жалпыланған шешімі келтіріліп, оның бірегейлігі дәлелденеді. Жалған домен әдісі арқылы алынған жуықталған модельдерге шешімдердің бар болуы және жинақтылығы теоремасы зерттеледі.

Түйін сөздер: жалған аймақтар әдісі, гидродинамика, океанология, тұтқыр сұйықтық, тұрақты емес домен, стационарлық есептер, бароклинді мұхит.

Introduction

The study of the processes that shape the global flow of seas and oceans, as well as the creation of mathematical models for analyzing the dynamics of the World Ocean and its regions, is an important area of modern research. Modeling is a key tool for understanding physical processes, analyzing the global spatial and temporal structure of the ocean, its interaction with the atmosphere, and regional variability in marine and ocean systems. Models also play an important role in the processing and assimilation of data from field observations [1-2]. The development of mathematical modeling of ocean dynamics, which has more than a century of experience, has led to a significant increase in understanding of the physical processes occurring in the marine environment, as well as improved methods and models for their analysis. In particular, the question of the relationship between barotropic and baroclinic components in the dynamics of sea currents, first posed in the mid-20th century by P.S. Lineikin and remaining significant to this day, remains relevant [3]. In addition, in recent years there has been increasing interest in studying the patterns of baroclinic fields, disturbances, and anomalies in the ocean, including the analysis of observational data, theoretical studies of the propagation of disturbances in a simplified oceanic environment, and numerical modeling. The study of ocean anomalies includes the analysis of remote connections and the formation of baroclinic and barotropic responses in areas remote from sources of external influences. For example, significant results have been obtained on the links between variability in the tropical

Pacific Ocean and variations in Antarctic ice extent. The physical mechanisms responsible for remote connections in the ocean and in the system of interaction with the atmosphere have been studied.

The concept of the baroclinic layer as a surface boundary layer located under the Ekman friction layer was a key idea in the studies of P.S. Lineikin [4]. Because the density of the upper ocean usually depends on temperature, the baroclinic layer is often assumed to coincide with the thermocline. However, such a comparison is not always accurate, given that seawater density is also influenced by salinity, especially in certain marine regions, such as the Black Sea, where density stratification is mainly determined by salinity. The basic principles of the theory of the baroclinic layer in the ocean can be derived from a complete set of primitive equations, including horizontal projections of the momentum balance equations, the hydrostatic equation, the mass conservation equation, the heat and salt diffusion equations, and the equation of state.

But these equations are nonlinear, even far from areas where jet streams are concentrated. Therefore, their analysis requires a simplification that preserves the key features of the phenomenon. P.S. Lineikin used linearized equations of motion and an equation of state that related the density of sea water to temperature and salinity through a linear dependence [5]. Under the assumption that the diffusion coefficients of heat and salt are identical, it is possible to replace the equations describing changes in temperature and salinity with a single equation for the diffusion of seawater density. However, this equation is still nonlinear and has no general solution. Further simplification is based on the hypothesis of the existence of an underlying density stratification that depends solely on the vertical coordinate. Assuming a constant vertical diffusion coefficient, the only way to create a stable base stratification is a linear increase in density with depth.

Our study focuses on the numerical simulation of the steady-state problem of baroclinic ocean motion, which is closely related to the practical problem of hydrodynamic forecasting.

Formulation of the problem

The stationary problem of the motion of a baroclinic ocean is reduced to solving the following system of equations

$$
\begin{aligned}
(\vec{v}\cdot\vec{\nabla})u &= \mu_0\frac{\partial^2 u}{\partial z^2}+\mu\Delta u-\frac{\partial p}{\partial x}-\ell v,\\
(\vec{v}\cdot\vec{\nabla})v &= \mu_0\frac{\partial^2 v}{\partial z^2}+\mu\Delta v-\frac{\partial p}{\partial y}+\ell u,
\end{aligned}
\tag{1}
$$

$$
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\qquad \frac{\partial p}{\partial z}=-\rho_0 g,\qquad \rho=a_0\theta+b_0,
$$

$$
\begin{gathered}
(\vec{v}\cdot\vec{\nabla})\theta=\lambda_0\frac{\partial^2\theta}{\partial z^2}+\lambda\Delta\theta+\vec{f},\\
\vec{v}=(u,v,w),\qquad \vec{u}=(u,v),\qquad \vec{\nabla}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right).
\end{gathered}
\tag{2}
$$

with boundary conditions

$$
\begin{gathered}
\left.\frac{\partial\vec{u}}{\partial z}\right|_{z=H}=\left.\frac{\partial\vec{u}}{\partial z}\right|_{z=0}=0,\qquad \vec{u}|_{\partial D_0}=0,\qquad \theta|_{z=0}=\theta|_{z=H}=0,\\
w|_{z=0}=w|_{z=H}=0,\qquad \theta|_{\partial D_0}=0,\qquad z\in[0,H].
\end{gathered}
\tag{3}
$$

In equations (1), (2), u, v are the projections of velocity on the x and y axes, respectively (the x and y axes are located in the horizontal plane); w – velocity projection onto the vertical (z axis directed downward); ρ – seawater density anomaly normalized to the average seawater density; p – pressure anomaly normalized to the average density of sea water.

Using the method given in [6], system (1), (2) can be reduced to the following form

$$(\vec{v}\cdot\vec{\nabla})u = \mu_0\frac{\partial^2 u}{\partial z^2} + \mu\Delta u - \frac{\partial\xi}{\partial x} - \ell v + \frac{\partial}{\partial x}\int_0^z \rho_0 g dz,$$

$$(\vec{v}\cdot\vec{\nabla})v = \mu_0\frac{\partial^2 v}{\partial z^2} + \mu\Delta v - \frac{\partial\xi}{\partial y} + \ell u + \frac{\partial}{\partial y}\int_0^z \rho_0 g dz,$$

$$\int_0^H \operatorname{div}\vec{u}dz = 0, \tag{4}$$

$$(\vec{v}\cdot\vec{\nabla})\theta = \lambda_0\frac{\partial^2\theta}{\partial z^2} + \lambda\Delta\theta + \vec{f}, \qquad \vec{v} = \left(u_1 v_1 - \int_0^z \operatorname{div}\vec{u}dz\right),$$

$$\frac{\partial\xi}{\partial z} = 0, \qquad \int_{D_0}\xi dxdy = 0. \tag{5}$$

Consider in the region $\Omega_2 = \Omega_0 \cup \Omega_1$ a problem with a small parameter

in Ω_0

$$(\vec{v}^\varepsilon\cdot\vec{\nabla})u^\varepsilon = \mu_0\frac{\partial^2 u^\varepsilon}{\partial z^2} + \mu\Delta u^\varepsilon - \frac{\partial\xi^\varepsilon}{\partial x} - \ell v^\varepsilon + \frac{\partial}{\partial x}h(x,y,z,\theta^\varepsilon),$$

$$(\vec{v}^\varepsilon\cdot\vec{\nabla})v^\varepsilon = \mu_0\frac{\partial^2 v^\varepsilon}{\partial z^2} + \mu\Delta v^\varepsilon - \frac{\partial\xi^\varepsilon}{\partial y} + \ell u^\varepsilon + \frac{\partial}{\partial y}h(x,y,z,\theta^\varepsilon), \tag{6}$$

$$(\vec{v}^\varepsilon\cdot\vec{\nabla})\theta^\varepsilon = \lambda_0\frac{\partial^2\theta^\varepsilon}{\partial z^2} + \lambda\Delta\theta^\varepsilon + \vec{f},$$

in Ω_1

$$(\vec{v}^\varepsilon\cdot\vec{\nabla})u^\varepsilon = \mu_0\frac{\partial^2 u^\varepsilon}{\partial z^2} + \frac{\mu}{\varepsilon}\Delta u^\varepsilon - \frac{\partial\xi^\varepsilon}{\partial x},$$

$$(\vec{v}^\varepsilon\cdot\vec{\nabla})v^\varepsilon = \mu_0\frac{\partial^2 v^\varepsilon}{\partial z^2} + \frac{\mu}{\varepsilon}\Delta v^\varepsilon - \frac{\partial\xi^\varepsilon}{\partial y},$$

$$(\vec{v}^\varepsilon\cdot\vec{\nabla})\theta^\varepsilon = \lambda_0\frac{\partial^2\theta^\varepsilon}{\partial z^2} + \frac{\lambda}{\varepsilon}\Delta\theta^\varepsilon, \tag{7}$$

$$\int_0^H \operatorname{div}\vec{u}^\varepsilon dz = 0, \qquad \text{в } \Omega_2, \qquad \frac{\partial\xi^\varepsilon}{\partial z} = 0, \qquad \int_{D_2}\xi^\varepsilon dxdy = 0,$$

with terms of agreement

$$\vec{u}^\varepsilon|_{\partial D_0} = 0,$$

$$\left[\frac{1}{\varepsilon}\mu\frac{\partial\vec{u}^\varepsilon}{\partial\vec{n}} - \xi^\varepsilon\cdot\vec{n}\right]\Big|_{\partial D_0^-} = \left[\mu\frac{\partial\vec{u}^\varepsilon}{\partial\vec{n}} - \xi^\varepsilon\cdot\vec{n}\right]\Big|_{\partial D_0^+}, \tag{8}$$

$$\left.\frac{\partial \vec{u}^{\varepsilon}}{\partial z}\right|_{z=H}=\left.\frac{\partial \vec{u}^{\varepsilon}}{\partial z}\right|_{z=0}=0, \qquad \vec{u}^{\varepsilon}|_{\partial D_2}=0, \qquad z\in[0,H]$$

$$\theta^{\varepsilon}|_{\partial D_0}=0, \qquad \frac{1}{\varepsilon}\left.\frac{\partial \theta^{\varepsilon}}{\partial \vec{n}}\right|_{\partial D_0^-}=\left.\frac{\partial \theta^{\varepsilon}}{\partial \vec{n}}\right|_{\partial D_0^+}.$$

$$\theta^{\varepsilon}|_{z=0}=\theta^{\varepsilon}|_{z=H}=0, \;\; \theta^{\varepsilon}|_{\partial D_2}=0.$$

Definition 1. A generalized solution to problem (6)-(8) is a pair of functions $(\vec{u}^{\varepsilon},\theta^{\varepsilon})$ such that $\vec{u}^{\varepsilon}(x,y,z)\in \dot{V}_2^1(\Omega_2)$, $\theta^{\varepsilon}\in \dot{W}_2^1(\Omega_2)$, satisfying the integral identities

$$\int_{\Omega_2}\vec{u}^{\varepsilon}\left(\vec{v}^{\varepsilon}\vec{\nabla}\right)\vec{\varphi}dxdydz+\mu_0\int_{\Omega_2}\frac{\partial \vec{u}^{\varepsilon}}{\partial z}\frac{\partial \vec{\varphi}}{\partial z}dxdydz+$$
$$+\mu\int_{\Omega_0}\nabla\vec{u}^{\varepsilon}\nabla\vec{\varphi}dxdydz+\frac{\mu}{\varepsilon}\int_{\Omega_1}\nabla\vec{u}^{\varepsilon}\nabla\vec{\varphi}dxdydz+ \tag{9}$$
$$+\int_{\Omega_0}[\ell u^{\varepsilon}\varphi_1-\ell v^{\varepsilon}\varphi_1]\,dxdydz=\int_{\Omega_0}h\mathrm{div}\vec{\varphi}\,dxdydz,$$

$$\int_{\Omega_2}\theta^{\varepsilon}\left(\vec{v}^{\varepsilon}\cdot\vec{\nabla}\right)\psi dxdydz+\lambda_0\int_{\Omega_2}\frac{\partial\theta^{\varepsilon}}{\partial z}\frac{\partial\psi}{\partial z}dxdydz+\lambda\int_{\Omega_0}\nabla\theta^{\varepsilon}\nabla\psi dxdydz+$$
$$+\frac{\lambda}{\varepsilon}\int_{\Omega_1}\nabla\theta^{\varepsilon}\nabla\psi\,dxdydz=\int_{\Omega_0}f\psi\,dxdydz, \tag{10}$$

for any $\vec{\varphi}(x,y,z)\in \dot{V}_2^1(\Omega_2)$, $\psi(x,y,z)\in \dot{W}_2^1(\Omega_2)$.

Definition 2. A strong solution to problem (6)-(8) is the function $\vec{u}^{\varepsilon}(x,y,z)\in W_2^2(\Omega_i)\cap \dot{V}_2^1(\Omega_i)$, $\nabla\xi^{\varepsilon}(x,y,z)\in L_2(\Omega_i)$, $i=0{,}1$ and $\theta^{\varepsilon}(x,y,z)\in W_2^2(\Omega_i)\cap \dot{W}_2^1(\Omega_i)$, satisfying (6)-(8) almost everywhere.

Let us obtain a priori estimates for the solution of problem (6), (8). To do this, in (9), (10) we set $\vec{\varphi},\psi$ equal to $\vec{u}^{\varepsilon},\theta^{\varepsilon}$, respectively. We have

$$\mu_0\left\|\frac{\partial\vec{u}^{\varepsilon}}{\partial z}\right\|_{L_2(\Omega_2)}^2+\mu\|\nabla\vec{u}^{\varepsilon}\|_{L_2(\Omega_0)}^2+\frac{\mu}{\varepsilon}\|\nabla\vec{u}^{\varepsilon}\|_{L_2(\Omega_1)}^2+$$
$$+\lambda_0\left\|\frac{\partial\theta^{\varepsilon}}{\partial z}\right\|_{L_2(\Omega_2)}^2+\lambda\|\nabla\theta^{\varepsilon}\|_{L_2(\Omega_0)}^2+\frac{\lambda}{\varepsilon}\|\nabla\theta^{\varepsilon}\|_{L_2(\Omega_1)}^2\leq \tag{11}$$
$$\leq c\left[\|h\|_{L_2(\Omega_0)}^2+\|f\|_{L_2(\Omega_0)}^2\right]=\mathbb{C}_1<\infty.$$

By virtue of the maximum principle for the elliptic equation we have

$$|\theta^{\varepsilon}|\leq M<\infty$$

Moreover, M does not depend on ε.

Next, we multiply the first two equations (6), (7) by $\frac{\partial}{\partial z}\left(z^2\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)$ scalarly in $L_2(\Omega_2)$. As a result of integration by parts we have

$$\int_{\Omega_2}\left(\vec{v}^\varepsilon\vec{\nabla}\right)\vec{u}^\varepsilon\frac{\partial}{\partial z}\left(z^2\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)dxdydz=\mu_0\int_{\Omega_2}\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}\left(2z\frac{\partial \vec{u}^\varepsilon}{\partial z}+z^2\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}\right)dxdydz+$$
$$+\mu\int_{\Omega_0}z^2\left(\frac{\partial\nabla\vec{u}^\varepsilon}{\partial z}\right)^2dxdydz+\frac{\mu}{\varepsilon}\int_{\Omega_1}z^2\left(\frac{\partial\nabla\vec{u}^\varepsilon}{\partial z}\right)dxdydz+ \tag{12}$$
$$+\int_{\Omega_2}\xi\frac{\partial}{\partial z}z^2\frac{\partial}{\partial z}\operatorname{div}\vec{u}^\varepsilon\,dxdydz+\int_{\Omega_2}\nabla h\left(2z\frac{\partial \vec{u}^\varepsilon}{\partial z}+z^2\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}\right)dxdydz.$$

Let us estimate the integrals on the right side as follows:

$$\int_{\Omega_2}\left(\mu_0\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}+\nabla h\right)\left(2z\frac{\partial \vec{u}^\varepsilon}{\partial z}+z^2\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}\right)dxdydz\geq$$
$$\geq\frac{\mu_0}{2}\int_{\Omega_2}z^2\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}dxdydz-c\left(\|\nabla h\|^2_{L_2(\Omega_2)}+\left\|\frac{\partial \vec{u}^\varepsilon}{\partial z}\right\|^2_{L_2(\Omega_2)}\right),$$

$$\int_{\Omega_2}\xi\frac{\partial}{\partial z}z^2\frac{\partial}{\partial z}\operatorname{div}\vec{u}^\varepsilon\,dxdydz=0.$$

Now let's evaluate the terms on the left side. Given the ratio

$$\frac{\partial \vec{u}^\varepsilon}{\partial z}\cdot\frac{\partial}{\partial z}\left(z^2\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)=\frac{\partial}{\partial z}\left(z^2\left(\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)^2\right)-\frac{z^2}{2}\frac{\partial}{\partial z}\left(\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)^2,$$

we have

$$\int_{\Omega_2}\int_0^z\left(u_x^\varepsilon+v_y^\varepsilon\right)dz\frac{\partial \vec{u}^\varepsilon}{\partial z}\cdot\frac{\partial}{\partial z}\left(z^2\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)dxdydz=$$
$$=\int_{\Omega_2}z\int_0^z\left(u_x^\varepsilon+v_y^\varepsilon\right)dz\left(\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)^2dxdydz-\frac{1}{2}\int_{\Omega_2}\left(u_x^\varepsilon+v_y^\varepsilon\right)z^2\left(\frac{\partial \vec{u}^\varepsilon}{\partial z}\right)^2dxdydz\leq$$
$$\leq c\left(\left\|\frac{1}{z}\int_0^z\left(u_x^\varepsilon+v_y^\varepsilon\right)dz\right\|_{L_2(\Omega_2)}+\|\nabla\vec{u}^\varepsilon\|_{L_2(\Omega_2)}\right)\left\|z\frac{\partial \vec{u}^\varepsilon}{\partial z}\right\|^2_{L_4(\Omega_2)}$$

Taking into account Hardy's inequality [7]

$$\left\|\frac{1}{z}\int_0^z\left(u_x^\varepsilon+v_y^\varepsilon\right)dz\right\|_{L_p(\Omega_2)}\leq c\left\|\left(u_x^\varepsilon+v_y^\varepsilon\right)\right\|_{L_p(\Omega_2)}$$

multiplicative

$$\left\|z\frac{\partial \vec{u}^\varepsilon}{\partial z}\right\|^2_{L_4(\Omega_2)}\leq c\left[\left\|z\frac{\partial \vec{u}^\varepsilon}{\partial z}\right\|_{L_2(\Omega_2)}\left\|z\frac{\partial}{\partial z}\vec{\nabla}\vec{u}^\varepsilon\right\|_{L_2(\Omega_2)}+\|\vec{\nabla}\vec{u}^\varepsilon\|_{L_2(\Omega_2)}\right],$$

and Young's inequality, we get

$$\int_{\Omega_2}\int_0^z (u_x + v_y)\,dz\frac{\partial\vec{u}^\varepsilon}{\partial z}\cdot\frac{\partial}{\partial z}\left(z^2\frac{\partial\vec{u}^\varepsilon}{\partial z}\right)dxdydz \leq$$

$$\leq \frac{\delta_1}{2}\left\|z\frac{\partial}{\partial z}\vec{\nabla}\vec{u}^\varepsilon\right\|^2_{L_2(\Omega_2)} + c\delta_1^{-1}\|\vec{\nabla}\vec{u}^\varepsilon\|^2_{L_2(\Omega_2)}\left\|z\frac{\partial\vec{u}^\varepsilon}{\partial z}\right\|^2_{L_2(\Omega_2)}$$

The remaining terms are evaluated similarly. Thus, from (12) it follows the estimate

$$\mu_0\int_{\Omega_2} z^2\left(\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}\right)^2 dxdydz + \mu\int_{\Omega_0}\left(\frac{\partial}{\partial z}\nabla\vec{u}^\varepsilon\right)^2 z^2 dxdydz +$$

$$+\frac{\mu}{\varepsilon}\int_{\Omega_1}\left(\frac{\partial}{\partial z}\nabla\vec{u}^\varepsilon\right)z^2 dxdydz \leq \mathbb{C}_2 < \infty.$$

Using it, we estimate the following quantity

$$\int_{\Omega_2}\left|(\vec{v}^\varepsilon\cdot\vec{\nabla})\vec{u}^\varepsilon\right|^2 dx \leq \|(\vec{u}^\varepsilon\cdot\nabla)\vec{u}^\varepsilon\|^2_{L_2(\Omega_2)} + \left\|\int_0^z (u_x^\xi + v_y^\xi)dz\cdot\frac{\partial\vec{u}^\varepsilon}{\partial z}\right\|^2_{L_2(\Omega_2)};$$

$$\left\|\frac{1}{z}\int_0^z (u_x^\xi + v_y^\xi)dz\right\|_{L_4(\Omega_2)} \leq C\left(\sum_{i=0}^{1}\left(\|\vec{u}^\varepsilon\|^{\frac{3}{2}}_{W_2^2(\Omega_i)} + \|\vec{u}^\varepsilon\|^{\frac{3}{2}}_{W_2^2(\Omega_i)}\cdot\left\|\vec{\nabla}z\frac{\partial\vec{u}^\varepsilon}{\partial z}\right\|^{\frac{3}{2}}_{L_2(\Omega_i)}\right)\|\vec{\nabla}\vec{u}^\varepsilon\|^2_{L_2(\Omega_i)}\right) \leq$$

$$\leq C\sum_{i=0}^{1}\|\vec{u}^\varepsilon\|^{\frac{3}{2}}_{W_2^2(\Omega_i)} \leq \delta\sum_{i=0}^{1}\|\vec{u}^\varepsilon\|^{\frac{3}{2}}_{W_2^2(\Omega_i)} + C\delta,$$

$$\|\vec{u}^\varepsilon\cdot\nabla\vec{u}^\varepsilon\|^2_{L_2(\Omega_2)} \leq \|\vec{u}^\varepsilon\|^2_{L_4(\Omega_2)}\cdot\|\nabla\vec{u}^\varepsilon\|^2_{L_4(\Omega_2)} \leq C\sum_{i=0}^{1}\|\vec{u}^\varepsilon\|^2_{L_4(\Omega_i)}\cdot\|\nabla\vec{u}^\varepsilon\|^2_{L_4(\Omega_i)} \leq$$

$$\leq C\sum_{i=0}^{1}\left(\|\nabla\vec{u}^\varepsilon\|_{L_2(\Omega_i)}\cdot\|\vec{u}^\varepsilon\|^{\frac{3}{2}}_{W_2^2(\Omega_i)} + \|\nabla\vec{u}^\varepsilon\|^2_{L_2(\Omega_i)}\right) \leq \delta\sum_{i=0}^{1}\|\vec{u}^\varepsilon\|^2_{W_2^2(\Omega_i)} + C_3\delta.$$

The obtained relations for sufficiently small δ and Theorem 1 from [1] allow us to establish the inequality

$$\left\|\frac{\partial^2\vec{u}^\varepsilon}{\partial z^2}\right\|_{L_2(\Omega_2)} + \|\nabla^2\vec{u}^\varepsilon\|_{L_2(\Omega_0)} + \frac{1}{\varepsilon}\|\nabla^2\vec{u}^\varepsilon\|_{L_2(\Omega_1)} \leq C\left(\left\|(\vec{v}^\varepsilon\cdot\vec{\nabla})\vec{u}^\varepsilon\right\|_{L_2(\Omega_2)} + \|\vec{h}\|_{L_2(\Omega_2)}\right) \leq C_4 \qquad (13)$$

Estimates (12) and (13) allow us to prove the following theorem using the Galerkin method.

Theorem 1.

A) Let $f(x,y,z) \in W_2^{-1}(\Omega_0), \rho, q$ be such that $\vec{h}(x,y,z,\varphi)$ belongs to $L_2(\Omega_0)$, if $\varphi \in L_2(\Omega_0)$. Then problem (6)-(8) has at least one generalized solution and estimates (11) and

$$\left\|\frac{\partial \theta^\varepsilon}{\partial z}\right\|_{L_2(\Omega_2)} + \|\nabla \theta^\varepsilon\|_{L_2(\Omega_0)} + \frac{1}{\varepsilon}\|\nabla \theta^\varepsilon\|_{L_2(\Omega_1)} \le C_5 \tag{14}$$

B) If $f(x,y,z) \in L_2(\Omega_0)$, $\partial\Omega_0, \partial\Omega_2 \subset C^2$ the functions ρ, q are such that $\nabla h(x,y,z,\theta^\varepsilon,\varphi) \in L_2(\Omega_2)$ if $\varphi \in W_2^1(\Omega_1)$. Then the generalized solution to problem (6)-(8) is strong and estimates (13) and

$$\left\|\frac{\partial^2 \theta^\varepsilon}{\partial z^2}\right\|_{L_2(\Omega_2)} + \|\nabla^2 \theta^\varepsilon\|_{L_2(\Omega_0)} + \frac{1}{\varepsilon}\|\nabla^2 \theta^\varepsilon\|_{L_2(\Omega_1)} \le C_6 \tag{15}$$

C) Let $f(x,y,z), h(x,y,z)$ be sufficiently small. Then the following estimate holds

$$\|\vec{u}^\varepsilon - \vec{u}\|_{W_2^1(\Omega_0)} + \|\theta^\varepsilon - \theta\|_{W_2^1(\Omega_1)} \le C_7 \tag{16}$$

Proof. The first two points of the theorem are a consequence of estimates (11), (13) and the theory of boundary value problems for parabolic equations.

From estimates (11), (13)-(15) it follows that from the sequence $\vec{u}^\varepsilon, \theta^\varepsilon$ we can select a sequence $\vec{u}_k^\varepsilon, \theta_k^\varepsilon$ strongly converging in $\dot{V}_2^1(\Omega_0)$ and $\dot{W}_2^1(\Omega_0)$ respectively. Passing to the limit in identities (9), (10), written for $\vec{u}_k^\varepsilon, \theta_k^\varepsilon$, we establish that the limit of this sequence is the solution of problem (3)-(5).

Let us estimate the speed of convergence. To do this, assume that different values of the small parameter $\varepsilon_1, \varepsilon_2$ correspond to the solutions $\vec{u}^{\varepsilon_1}, \theta^{\varepsilon_1}$ and $\vec{u}^{\varepsilon_2}, \theta^{\varepsilon_2}$. Let us introduce the notation $\vec{u}^{\varepsilon_1} - \vec{u}^{\varepsilon_2} = \vec{W}^\varepsilon$, $\theta^{\varepsilon_1} - \theta^{\varepsilon_2} = \eta^\varepsilon$, $\vec{v}^{\varepsilon_1} - \vec{v}^{\varepsilon_2} = \vec{w}^\varepsilon$.

Relations (9), (10) lead to identities for the functions $\vec{w}^\varepsilon, \eta^\varepsilon$

$$\begin{gathered}\int_{\Omega_2}\left[-\vec{u}^{\varepsilon_2}(\vec{w}^\varepsilon\cdot\vec{\nabla})\vec{\varphi} - \vec{W}^\varepsilon(\vec{v}^{\varepsilon_1}\cdot\vec{\nabla})\varphi + \mu_0\frac{\partial \vec{W}^\varepsilon}{\partial z}\cdot\frac{\partial\vec{\varphi}}{\partial z}\right]dxdydz + \mu\int_{\Omega_0}\nabla\vec{W}^\varepsilon\nabla\vec{\varphi}dxdydz + \\ +\mu\int_{\Omega_1}\left(\frac{\nabla\vec{u}^{\varepsilon_1}}{\varepsilon_1} - \frac{\nabla\vec{u}^{\varepsilon_2}}{\varepsilon_2}\right)\nabla\vec{\varphi}dxdydz + \int_{\Omega_0}\left(\ell\cdot\vec{W},\vec{\varphi}\right)dxdydz \\ = \int_{\Omega_0}[h(\theta^{\varepsilon_2}) - h(\theta^{\varepsilon_1})]\operatorname{div}\vec{\varphi}\,dxdydz.\end{gathered} \tag{17}$$

$$\begin{gathered}\int_{\Omega_2}\left[-\theta^{\varepsilon_2}(\vec{w}\cdot\vec{\nabla})\psi - \eta^\varepsilon(\vec{v}^{\varepsilon_1}\cdot\vec{\nabla})\psi + \lambda_0\frac{\partial\eta^\varepsilon}{\partial z}\cdot\frac{\partial\psi}{\partial z}\right]dxdydz + \\ +\lambda\int_{\Omega_0}\nabla\eta^\varepsilon\nabla\psi\,dxdydz + \lambda\int_{\Omega_1}\left(\frac{\nabla\theta^{\varepsilon_1}}{\varepsilon_1} - \frac{\nabla\theta^{\varepsilon_2}}{\varepsilon_2}\right)\nabla\psi dxdydz = 0.\end{gathered} \tag{18}$$

Let us put in (17), (18) $\vec{\varphi} = \vec{W}^\varepsilon, \psi = \eta^\varepsilon$. As a result we have

$$\mu_0\left\|\frac{\partial\vec{W}^\varepsilon}{\partial z}\right\|^2_{L_2(\Omega_2)}+\mu\left\|\nabla\vec{W}^\varepsilon\right\|^2_{L_2(\Omega_0)}\le\left|\int_{\Omega_2}\vec{u}^{\varepsilon_2}\left(\vec{w}^\varepsilon\cdot\vec{\nabla}\right)\vec{W}^\varepsilon dxdydz\right|+ \tag{19}$$

$$+\max_{(x,y,z)\in\Omega_0}\left|h'[\alpha\theta^{\varepsilon_1}+(1-\alpha)\theta^{\varepsilon_2}]\right|\cdot\|\eta^\varepsilon\|_{L_2(\Omega_0)}\left\|\mathrm{div}\vec{W}^\varepsilon\right\|_{L_2(\Omega_0)}+$$

$$+\left|\int_{\Omega_1}[h(\theta^{\varepsilon_1})-h(\theta^{\varepsilon_2})]\,\mathrm{div}(\vec{u}^{\varepsilon_1}-\vec{u}^{\varepsilon_2})dxdydz\right|,$$

$$\lambda_0\left\|\frac{\partial\eta^\varepsilon}{\partial z}\right\|_{L_2(\Omega_2)}+\lambda\left\|\vec{\nabla}\eta^\varepsilon\right\|^2_{L_2(\Omega_0)}\le\left|\int_{\Omega_2}\theta^{\varepsilon_2}\left(\vec{w}^\varepsilon\cdot\vec{\nabla}\right)\eta^\varepsilon dxdydz\right| \tag{20}$$

Let us evaluate the integrals on the right side of (19), (20).

$$\int_{\Omega_2}\vec{u}^{\varepsilon_2}\left(\vec{w}^\varepsilon\cdot\vec{\nabla}\right)\vec{W}^\varepsilon dxdydz\le C\max_{\Omega_2}|\vec{u}^{\varepsilon_2}|\cdot\left\|\vec{\nabla}\cdot\vec{W}^\varepsilon\right\|^2_{L_2(\Omega_2)}$$

$$\int_{\Omega_0}\theta^{\varepsilon_2}\left(\vec{w}^\varepsilon\cdot\vec{\nabla}\right)\eta^\varepsilon dxdydz\le C\max_{\Omega_2}|\theta^{\varepsilon_2}|\cdot\left\|\vec{\nabla}\eta^\varepsilon\right\|_{L_2(\Omega_0)}\cdot\left\|\vec{W}^\varepsilon\cdot\vec{\nabla}\right\|\le$$

$$\le\frac{\delta_2}{2}\left\|\vec{\nabla}\eta^\varepsilon\right\|^2_{L_2(\Omega_0)}+\frac{\delta_3^{-1}}{2}\max_{\Omega_2}|\theta^{\varepsilon_2}|\cdot\left\|\vec{\nabla}\cdot\vec{W}^\varepsilon\right\|^2_{L_2(\Omega_0)}.$$

$$\int_{\Omega_2}\vec{u}^{\varepsilon_1}\left(\vec{w}^\varepsilon\cdot\vec{\nabla}\right)\vec{W}^\varepsilon dxdydz=\int_{\Omega_2}\vec{u}^{\varepsilon_1}[(\vec{u}^{\varepsilon_1}-\vec{u}^{\varepsilon_2})\cdot\nabla]\,(\vec{u}^{\varepsilon_1}-\vec{u}^{\varepsilon_2})dxdydz+$$

$$+\int_{\Omega}\vec{u}^{\varepsilon_2}\int_0^z\left(\mathrm{div}\vec{W}\right)dz\cdot\frac{\partial\vec{W}}{\partial z}dxdydz;$$

$$\int_{\Omega_2}\vec{u}^{\varepsilon_1}\left((\vec{u}^{\varepsilon_1}-\vec{u}^{\varepsilon_2})\cdot\nabla\right)(\vec{u}^{\varepsilon_1}-\vec{u}^{\varepsilon_2})dxdydz\le\max_{\Omega_2}|\vec{u}^{\varepsilon_1}|\left\|\vec{W}\right\|_{L_2(\Omega_2)}\left\|\nabla\vec{W}\right\|_{L_2(\Omega_2)};$$

$$\int_{\Omega_2}\vec{u}^{\varepsilon_2}\int_0^z\left(\mathrm{div}\vec{W}\right)dz\cdot\frac{\partial\vec{W}}{\partial z}dxdydz\le\max_{\Omega_2}|\vec{u}^{\varepsilon_2}|\,C\left\|\nabla\vec{W}\right\|^2_{L_2(\Omega_2)};$$

$$\int_{\Omega_2}[h(\theta^{\varepsilon_1})-h(\theta^{\varepsilon_2})]\mathrm{div}[\vec{u}^{\varepsilon_1}-\vec{u}^{\varepsilon_2}]dxdydz\le$$

$$\le C\left(\max|h'|\|\theta^{\varepsilon_1}-\theta^{\varepsilon_2}\|_{L_2(\Omega_1)}\right)\cdot\left\|\nabla\vec{W}\right\|_{L_2(\Omega_2)}.$$

$$\int_{\Omega_2}\theta^{\varepsilon_2}\left(\vec{w}^\varepsilon\cdot\vec{\nabla}\right)\eta^\varepsilon dxdydz=\int_{\Omega_2}\theta^{\varepsilon_2}[(\vec{u}^{\varepsilon_1}-\vec{u}^{\varepsilon_2})\cdot\nabla](\theta^{\varepsilon_1}-\theta^{\varepsilon_2})\,dxdydz+$$

$$\int_{\Omega_2}\theta^{\varepsilon_2}\left[\int_0^z(\mathrm{div}\,\vec{u}^{\varepsilon_1}-\mathrm{div}\,\vec{u}^{\varepsilon_2})\,dz\right]\frac{\partial\eta^\varepsilon}{\partial z}dxdydz;$$

$$\left|\int_{\Omega_2} (\theta^{\varepsilon_2}(\vec{u}^{\varepsilon_1} - \vec{u}^{\varepsilon_2})\nabla)\,(\theta^{\varepsilon_1} - \theta^{\varepsilon_2})dxdydz\right| \le \max_{\Omega_2}|\theta^{\varepsilon_2}|\,\|\vec{u}^{\varepsilon_1} - \vec{u}^{\varepsilon_2}\|_{L_2(\Omega_2)}\,\|\nabla\eta\|_{L_2(\Omega_2)};$$

$$\left|\int_{\Omega_2} \theta^{\varepsilon_2}\left[\int_0^z (\operatorname{div}\vec{u}^{\varepsilon_1} - \operatorname{div}\vec{u}^{\varepsilon_2})\,dz\right]\frac{\partial\eta}{\partial z}dxdydz\right| \le \max_{\Omega_2}|\theta^{\varepsilon_2}|\,\left\|\nabla\overrightarrow{W}\right\|_{L_2(\Omega_2)}\|\nabla\eta\|_{L_2(\Omega_2)};$$

$$\int_{\Omega_2} \left(\ell\cdot\overrightarrow{W},\overrightarrow{W}\right)dxdydz = 0. \qquad (21)$$

$$\mu\int_{\Omega_1}\left(\frac{\nabla\vec{u}^{\varepsilon_1}}{\varepsilon_1} - \frac{\nabla\vec{u}^{\varepsilon_2}}{\varepsilon_2}, \nabla\overrightarrow{W}\right)dx = \mu\int_{\Omega_2}\left(\nabla\frac{\vec{u}^{\varepsilon_1} - \vec{u}^{\varepsilon_2}}{\varepsilon_1} + \frac{\nabla\vec{u}^{\varepsilon_2}}{\varepsilon_2} + \frac{\nabla\vec{u}^{\varepsilon_1}}{\varepsilon_1}, \nabla\overrightarrow{W}\right)dxdydz \ge$$

$$\ge \frac{1}{\varepsilon_1}\left\|\nabla\overrightarrow{W}\right\|^2_{L_2(\Omega_1)} - \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)\|\nabla\vec{u}^{\varepsilon_2}\|_{L_2(\Omega_1)}\left\|\nabla\overrightarrow{W}\right\|_{L_2(\Omega_1)} \ge$$

$$\ge \frac{1}{\varepsilon_1}\left\|\nabla\overrightarrow{W}\right\|^2_{L_2(\Omega_1)} - \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1\varepsilon_2}\cdot C\|\nabla\vec{u}^{\varepsilon_2}\|(\|\nabla\vec{u}^{\varepsilon_1}\| + \|\nabla\vec{u}^{\varepsilon_2}\|) \ge \qquad (22)$$

$$\ge \frac{1}{\varepsilon_2}\left\|\nabla\overrightarrow{W}\right\|^2_{L_2(\Omega_2)} - \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1\varepsilon_2}\cdot C\cdot\varepsilon_2(\varepsilon_1 + \varepsilon_2).$$

As a result of the obtained estimates (21), (22), for small data of the problem in the norm W_2 has the estimate

$$\left\|\overrightarrow{W}\right\|_{W_2^1(\Omega_2)} \le C\frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1\varepsilon_2}\cdot\varepsilon_2(\varepsilon_1 + \varepsilon_2)$$

from which follows (16) for ε_1 and ε_2 of the order of ε.

Remark 1. Similarly, we can study the boundary value problem for system (1), when at $z = 0$ the condition $\vec{u}|_{z=0} = 0$ is set.

Conclusion

In the course of the study, the fictitious domain method for a nonlinear stationary problem was considered

$$\begin{gathered}\left(\vec{v}^{\varepsilon}\cdot\vec{\nabla}\right)\vec{u}^{\varepsilon} = \mu_0\frac{\partial^2\vec{u}^{\varepsilon}}{\partial z^2} + \mu\Delta\vec{u}^{\varepsilon} - \nabla\xi^{\varepsilon} + \left[\vec{\ell}\cdot\vec{u}^{\varepsilon}\right]v^{\varepsilon} + \nabla h(x,y,z,\theta^{\varepsilon}),\text{ в }\Omega_0\\ \left(\vec{v}^{\varepsilon}\cdot\vec{\nabla}\right)\theta = \lambda_0\frac{\partial^2\theta^{\varepsilon}}{\partial z^2} + \lambda\Delta\theta^{\varepsilon} + \vec{f}\end{gathered} \qquad (23)$$

$$\begin{gathered}\left(\vec{v}^{\varepsilon}\cdot\vec{\nabla}\right)\vec{u}^{\varepsilon} = \mu_0\frac{\partial^2\vec{u}^{\varepsilon}}{\partial z^2} + \frac{1}{\varepsilon}\mu\Delta\vec{u}^{\varepsilon} - \nabla\xi^{\varepsilon},\text{ в }\Omega_1\\ \left(\vec{v}^{\varepsilon}\cdot\vec{\nabla}\right)\theta^{\varepsilon} = \lambda_0\frac{\partial^2\theta^{\varepsilon}}{\partial z^2} + \frac{1}{\varepsilon}\lambda\Delta\theta^{\varepsilon}\end{gathered} \qquad (24)$$

$$\int_0^H \operatorname{div}\,\vec{u}^{\varepsilon}dz = 0, \qquad \int_{D_0}\xi^{\varepsilon}dxdy = 0,\ \frac{\partial\xi^{\varepsilon}}{\partial z} = 0\ \text{ в }\Omega_2$$

subject to agreement

$$[\vec{u}^{\varepsilon}]|_{\partial D_0} = 0, \qquad z \in (0, H), \tag{25}$$

$$\left[\frac{1}{\varepsilon}\mu\frac{\partial\vec{u}^{\varepsilon}}{\partial n} - \xi^{\varepsilon}\cdot\vec{n}\right]\Bigg|_{\partial D_0^-} = \left[\mu\frac{\partial\vec{u}^{\varepsilon}}{\partial n} - \xi^{\varepsilon}\cdot\vec{n}\right]\Bigg|_{\partial D_0^+} \tag{26}$$

$$\frac{\partial\vec{u}^{\varepsilon}}{\partial z}\bigg|_{z=0} = \frac{\partial\vec{u}^{\varepsilon}}{\partial z}\bigg|_{z=H} = 0, \qquad \vec{u}^{\varepsilon}|_{\partial D_2} = 0, \qquad \frac{1}{\varepsilon}\frac{\partial\theta^{\varepsilon}}{\partial\vec{n}}\bigg|_{\partial D_0^-} = \frac{\partial\theta^{\varepsilon}}{\partial\vec{n}}\bigg|_{\partial D_0^+}, \qquad [\theta^{\varepsilon}]_{\partial D_0} = 0,$$

were

$$\vec{v}^{\varepsilon} = \left(u^{\varepsilon}, v^{\varepsilon}, -\int_0^z \operatorname{div}\vec{u}^{\varepsilon}dz\right), \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \qquad \vec{\ell} = (\ell, \ell).$$

The existence theorem for a strong solution to problem (25), (26) for system (23), (24) and the estimate

$$\|\vec{u}^{\varepsilon} - \vec{u}\|_{W_2^1(\Omega_0)} + \|\theta^{\varepsilon} - \theta\|_{W_2^1(\Omega_0)} \le C_{\varepsilon}$$

where $\vec{u}, \theta$ is the solution to the problem

$$\vec{u}|_{\partial\Omega_0} = 0, \qquad \theta|_{\partial\Omega_0} = 0$$

for a system of equations of the form (23).

Список использованных источников:

1 Линейкин П.С. Основные вопросы динамической теории бароклинного слоя моря. Л.: Гидрометеоиздат, 1957. - 139 с.

2 Марчук Г.И. Численные решения задачи динамики атмосферы и океана. – Л.: Гидрометеоиздат, 1974. – 303с.

3 Линейкин П.С. Об определении толщины бароклинного слоя моря // Докл. АН СССР. 1955а., Т. 101. № 3. С. 461−464.

4 Линейкин П.С. К динамике установившихся течений и неоднородном море // Докл. АН СССР., 1955б. Т. 105. № 6. С. 1215−1217.

5 Линейкин П.С. О ветровых течениях и бароклинном слое в море // Тр. ГОИН. 1955в. Вып. 29(41), С. 34−64.

6 Akhmetova, O., Issayev, S. Application of the fictitious region method to solving a model oceanology problem. Bulletin of Physics & Mathematical Sciences. 83, 3 (Sep. 2023), 34–42. DOI: https://doi.org/10.51889/2959-5894.2023.83.3.004

7 Никольский С.М. Приближение функций многих переменных и теоремы вложения. – М.: Наука, 1977. – 455с.

8 P. Sun and C. Wang, Distributed lagrange multiplier/fictitious domain finite element method for Stokes/parabolic interface problems with jump coefficients. Journal of Applied Numerical Mathematics, vol. 152, no. 4, pp. 199–220, 2020.

9 P. Sun, "Fictitious domain finite element method for Stokes/elliptic interface problems with jump coefficients," Journal of Computational and Applied Mathematics, vol. 356, no. 3, pp. 81–97, 2019.

10 Daniele Boffi, Fabio Credali, Lucia Gastaldi. On the interface matrix for fluid–structure interaction problems with fictitious domain approach. Journal of Computer Methods in Applied Mechanics and Engineering, vol.401, Part B, 1 November 2022, 115650, https://doi.org/10.1016/j.cma.2022.115650

11 Mikihiro Tajima, Takayuki Yamada. Topology optimization with geometric constraints for additive manufacturing based on coupled fictitious physical model. Journal of Computer Methods in Applied

Mechanics and Engineering, Volume 417, Part A, 1 December 2023, 116415, https://doi.org/10.1016/j.cma.2023.116415

12 Ziyang Huang, Guang Lin, Arezoo M. Ardekani. A consistent and conservative Phase-Field method for multiphase incompressible flows. Journal of Computational and Applied Mathematics, Volume 408, July 2022, 114116. https://doi.org/10.1016/j.cam.2022.114116

References:

1 Linejkin P.S. (1957) Osnovnye voprosy dinamicheskoj teorii baroklinnogo sloja morja. [Basic questions of the dynamic theory of the baroclinic layer of the sea.] L.: Gidrometeoizdat 139 s. (in Russian)

2 Marchuk G.I. (1974) Chislennye reshenija zadachi dinamiki atmosfery i okeana. [Numerical solutions to the problem of atmospheric and ocean dynamics] L.: Gidrometeoizdat, 303 s. (in Russian)

3 Linejkin P.S. (1955) Ob opredelenii tolshhiny baroklinnogo sloja morja [On determining the thickness of the baroclinic layer of the sea] Dokl. AN SSSR. T. 101. № 3. S. 461−464. (in Russian)

4 Linejkin P.S. (1955) K dinamike ustanovivshihsja techenij i neodnorodnom more [On the dynamics of steady currents and a heterogeneous sea] Dokl. AN SSSR., T. 105. № 6. S. 1215−1217. (in Russian)

5 Linejkin P.S. (1955) O vetrovyh techenijah i baroklinnom sloe v more [On wind currents and the baroclinic layer in the sea] Tr. GOIN. Vyp. 29(41), S. 34−64. (in Russian)

6 Akhmetova, O., Issayev, S. Application of the fictitious region method to solving a model oceanology problem. Bulletin of Physics & Mathematical Sciences. 83, 3 (Sep. 2023), 34–42. DOI: https://doi.org/10.51889/2959-5894.2023.83.3.004

7 Nikol'skij S.M. (1977) Priblizhenie funkcij mnogih peremennyh i teoremy vlozhenija. [Approximation of functions of several variables and embedding theorems] M.: Nauka, 455s. (in Russian)

8 P. Sun and C. Wang, Distributed Lagrange multiplier/fictitious domain finite element method for Stokes/parabolic interface problems with jump coefficients. Journal of Applied Numerical Mathematics, vol. 152, no. 4, pp. 199–220, 2020.

9 P. Sun, "Fictitious domain finite element method for Stokes/elliptic interface problems with jump coefficients," Journal of Computational and Applied Mathematics, vol. 356, no. 3, pp. 81–97, 2019.

10 Daniele Boffi, Fabio Credali, Lucia Gastaldi. On the interface matrix for fluid–structure interaction problems with fictitious domain approach. Journal of Computer Methods in Applied Mechanics and Engineering, vol.401, Part B, 1 November 2022, 115650, https://doi.org/10.1016/j.cma.2022.115650

11 Mikihiro Tajima, Takayuki Yamada. Topology optimization with geometric constraints for additive manufacturing based on coupled fictitious physical model. Journal of Computer Methods in Applied Mechanics and Engineering, Volume 417, Part A, 1 December 2023, 116415, https://doi.org/10.1016/j.cma.2023.116415

12 Ziyang Huang, Guang Lin, Arezoo M. Ardekani. A consistent and conservative Phase-Field method for multiphase incompressible flows. Journal of Computational and Applied Mathematics, Volume 408, July 2022, 114116. https://doi.org/10.1016/j.cam.2022.114116