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MODELING OF PHYSICAL PROCESSES AND MECHANICAL SYSTEMS

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N.A. Beissen[1], A. Muratkhan[1*], T. Kapar[2], K. Amanbayeva[2], T. Yernazarov[2]
[1]Al-Farabi Kazakh National University, Almaty, Kazakhstan
[2]Kazakh National Women's Teacher Training University, Almaty, Kazakhstan
[*]e-mail: arai12@mail.ru

THE INTERNAL STRUCTURE AND GENERAL RELATIVISTIC CALCULATIONS OF COMPACT OBJECTS

Abstract

In this study, we examine dense compact objects, such as white dwarfs and neutron stars, through the lens of Einstein's theory of gravity. Our focus is on understanding these objects when they are not perfectly spherical, using a mathematical description for their gravitational fields. We consider the quadrupole moment as an additional parameter that explicitly enters the equilibrium equations and the geometry of spacetime. In fact, most studies of equilibrium conditions in relativistic astrophysics are limited to the case of spherically symmetric sources. We construct approximate interior and exterior line elements, considering the quadrupole moment up to the first order, to describe static deformed compact objects. We pay particular attention to the interiors of slightly deformed compact objects, applying a specific formula known as the equation of state (EoS). This critical component enables us to understand how these stars balance the force of their own gravity with internal pressure. The EoS is pivotal in determining how matter behaves under the extreme conditions of density and pressure found within these compact objects.

Keywords: gravitational field, equations of state, compact objects, quadrupole moment.

Н.А. Бейсен[1], А. Муратхан[1], Т. Капар[2], К. Аманбаева[2], Т. Ерназаров[1]
[1]Казахский национальный университет имени Аль-Фараби, Казахстан, г. Алматы
[2]Казахский национальный женский педагогический университет, Казахстан, г. Алматы

ВНУТРЕННЯЯ СТРУКТУРА И ОБЩЕРЕЛЯТИВИСТСКИЕ РАСЧЕТЫ КОМПАКТНЫХ ОБЪЕКТОВ

Аннотация

В исследовании мы рассматриваем плотные компактные объекты, такие как белые карлики и нейтронные звезды, через призму теории гравитации Эйнштейна. Наша цель – понять эти объекты, когда они не являются идеально сферическими, используя математическое описание их гравитационных полей. Мы учитываем квадрупольный момент как дополнительный параметр, который явно входит в уравнения равновесия и геометрию пространства-времени. Фактически, большинство исследований условий равновесия в релятивистской астрофизике ограничивается случаем сферически симметричных источников. Мы строим приблизительные внутренние и внешние элементы линии, учитывая квадрупольный момент до первого порядка, чтобы описать статически деформированные компактные объекты. Мы уделяем особое внимание внутренностям слегка деформированных компактных объектов, применяя определенную формулу, известную как уравнение состояния (EoS). Этот критический компонент позволяет нам понять, как эти звезды уравновешивают силу собственной гравитации с внутренним давлением. EoS имеет решающее значение для определения поведения материи в экстремальных условиях плотности и давления, которые встречаются в этих компактных объектах.

Ключевые слова: гравитационное поле, уравнения состояния, компактные объекты, квадрупольный момент.

Н.А. Бейсен[1], А. Муратхан[1], Т. Қапар[2], К. Аманбаева[2], Т. Ерназаров[1]
[1]Әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қ., Қазақстан
[2]Қазақ ұлттық қыздар педагогикалық университеті, Алматы қ., Қазақстан

ЫҚШАМ ОБЪЕКТІЛЕРДІҢ ІШКІ ҚҰРЫЛЫМЫ ЖӘНЕ ЖАЛПЫ РЕЛЯТИВИСТІК ЕСЕПТЕУЛЕРІ

Аңдатпа

Бұл жұмыста, Эйнштейннің гравитация теориясы негізінде ақ ергежейлі және нейтрондық жұлдыздар сияқты тығыз жинақы нысандарды қарастырамыз. Біздің мақсатымыз - бұл объектілердің гравитациялық өрістерінің математикалық сипаттамасын пайдалана отырып, олар сфералық симметриялы болмаған жағдайда олардың физикасын түсіну. Тепе-теңдік теңдеулері мен кеңістік-уақыт геометриясына нақты енгізілген қосымша параметр ретінде квадрупольдік моментті ескереміз. Шын мәнінде, релятивистік астрофизикадағы тепе-теңдік жағдайларын зерттеудің көпшілігі сфералық симметриялы көздермен шектеледі. Статикалық деформацияланған ықшам нысандарды сипаттау үшін бірінші ретке дейінгі квадрупольдік моментті ескере отырып, жуықталған ішкі және сыртқы сызық элементтерін пайдаланамыз. Күй теңдеуілері үшін (КТ) арнайы формуланы қолдану арқылы мардымсыз деформацияланған ықшам нысандардың ішкі құрылымына назар аударамыз. Бұл маңызды компонент жұлдыздардың өздерінің гравитациялық тартылыс күшін ішкі қысыммен қалай теңестіретінін түсінуге мүмкіндік береді. КТ осы ықшам нысандарда кездесетін тығыздық пен қысымның төтенше жағдайларындағы материяның әрекетін анықтау үшін өте маңызды.

Түйін сөздер: гравитациялық өріс, күй теңдеулері, ықшам объектілер, квадрупольдік момент.

Introduction

Studying the universe's dense objects, such as white dwarfs and neutron stars, is fundamental in the exploration of space using Einstein's general theory of relativity. These dense objects range from planets to black holes and are characterized by their significant mass relative to their size. Understanding these objects is crucial for unraveling the mysteries of the cosmos.

In the framework of general relativity, accurately describing the gravitational field in any scenario involving gravity is vital. The metric tensor, which solves Einstein's equations, contains all details of the gravitational field. For compact objects like stars or planets, this entails analyzing both an interior and an exterior solution that encompasses mass, angular momentum, and the quadrupole moment. While the Kerr spacetime represents an exterior solution for objects with mass and angular momentum, discovering a compatible interior solution poses a significant challenge in general relativity. Despite various attempts to find such solutions, including the use of exotic matter and specific equations of state, a definitive solution remains elusive. This situation underscores the necessity of exploring alternative approaches and incorporating additional physical parameters for a more comprehensive description of the gravitational field [1,2].

Recent research efforts, utilizing tools like NICER and LOFT in the electromagnetic spectrum, or Advanced LIGO and the Einstein Telescope for gravitational waves, aim to investigate or constrain further details that reveal the structure of spinning compact objects, including their deformability, known as "Love numbers"[3]. The interaction between spin and orbit in binary pulsars might provide insights into the object's moment of inertia, and studying gravitational waves could reveal tidal Love numbers and shed light on the equation of state [4].

The behavior of compact objects can also be described by relativistic multipole moments. These moments can be determined through experiments involving gyroscopes or particles in orbit, linking various physical aspects of the source with the multipole moments of the surrounding spacetime, assuming a comprehensive solution is available.

The physics of compact objects involves three basic steps: the use of conservation laws representing hydrostatic relativistic equilibrium conditions, the construction of an appropriate equation of state, the matching with the exterior solution, and, finally, the comparison with observational data [5]. Our focus is on compact objects at the zero temperature, where the force of gravity plays a significant role. We aim to connect models of what's inside these objects, seen as either perfect or imperfect fluids in balance, with how they appear in the emptiness of space, factoring in their spin and shape changes. To tackle Einstein's field equations for these objects, we simplify by assuming they are not moving or changing shape, which makes the mathematical problems easier to solve. This method helps us understand the gravity around these unique space objects, underlining the difficulties and techniques in linking their inside with their outside.

Consider that the internal structure of compact objects features a mass distribution and is characterized by a quadrupole moment. In Einstein's theory, the Schwarzschild solution describes a spherically symmetric gravitational field in a vacuum, and according to Birkhoff's theorem, it is the only solution of its kind. Deviations from spherical symmetry are often explained using multipole moments, with the quadrupole moment being particularly significant. For axially symmetric mass distributions that include a quadrupole, there is no equivalent to Birkhoff's theorem, meaning the gravitational field can be represented by various metrics [5].

A notably simple metric was introduced in [6], suggesting the use of the Zipoy–Voorhees transformation [7,8] to create a quadrupolar vacuum solution. This metric has been referred to as the Zipoy–Voorhees metric, δ metric, γ metric, and q metric in various studies [7-14]. Solutions for interiors containing quadrupoles was discovered in [15], and a technique for producing perfect fluid quadrupolar solutions was developed in [16-17]. Approaches for approximate interior solutions and analyses of the exterior q metric's properties were explored in [18]. More recently, a study [17] examined six different extensions of the Schwarzschild metric that incorporate quadrupoles, identifying a shared characteristic where the hypersurface $r=2m$ is singular. It's likely that other precise solutions to Einstein's vacuum equations share these features [19-23].

To find the solutions for the gravitational field of interior structure of slightly deformed compact objects, we follow our previous study, where we embarked on a comprehensive examination of the internal structure of compact astrophysical objects, such as neutron stars and white dwarfs, through the theoretical framework of Einstein's general relativity and we will divide it into several important parts as follows:

Construct the field equations with the quadruple parameter. Building on our established method, we continue to explore the interior solutions to Einstein's equations by incorporating axially symmetric and static spacetimes. The quadrupole moment is considered to first order, serving as a fundamental parameter in our analysis of compact object deformation.

Choosing the appropriate EOSs, particularly relevant for white dwarfs and neutron stars. These EoS provide a more nuanced view of the internal pressures and densities, essential for realistic modeling.

Finding the matching conditions. A critical aspect of our methodology is ensuring that the interior solutions smoothly match the exterior spacetime, maintaining continuity and physical realism. This involves aligning the interior and exterior metrics at the surface of the compact object, a process that requires careful consideration of the boundary conditions dictated by the chosen EoS. The matching conditions are pivotal in validating the feasibility of the interior solutions, ensuring they correspond to observable properties of compact objects.

Numerical Solution of Field Equations: With these EoS as our foundation, we numerically solve the field equations to obtain the interior metrics of compact objects. This step involves detailed computational methods to handle the increased complexity introduced by the new EoS, aiming to find solutions that are both mathematically sound and physically plausible.

Research methodology

The line element and field equations

Numerous ways exist to express the corresponding line element, and theoretically, they should all be equivalent [5]. Nevertheless, specific expressions of the line element prove to be particularly useful for examining certain issues. Experience with numerical solutions for perfect fluids suggests that for the scenario in question, the line element for the axisymmetric case

$$ds^2 = f dt^2 - \frac{e^{2\gamma}}{f}\left(\frac{dr^2}{h} + d\theta^2\right) - \frac{\mu^2}{f} d\phi^2 , \tag{1}$$

here $f = f(r,\theta), \gamma = \gamma(r,\theta), \mu = \mu(r,\theta),$ and $h = h(r)$. A redefinition of the coordinate r leads to an equivalent line element which has been used to investigate anisotropic static fluids [18].

The Einstein equations for a perfect fluid with 4velocity U_α, density ρ, and pressure p (we use geometric units with $G = c = 1$)

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi\left[(\rho + p) U_\alpha U_\beta - p g_{\alpha\beta}\right] \tag{2}$$

for the line element (3) can be represented as two second order differential equations for μ and f and the function γ is determined by a set of two fist order partial differential equations. Notice also that the pressure p and the density ρ must be given *a priori* in order to solve the main set of differential equations for μ and f. In addition, from the conservation law

$$T^{\alpha\beta}_{;\beta} = 0 , \tag{3}$$

we obtain two first-order differential equations for the pressure

$$p_{,r} = -\frac{1}{2}(p + \rho)\frac{f_{,r}}{f} , \quad p_{,\theta} = -\frac{1}{2}(p + \rho)\frac{f_{,\theta}}{f} , \tag{4}$$

that can be integrated for any given functions $f(r,\theta)$ and $\rho(r,\theta)$, which satisfy Einstein's equations. Finding physically viable solutions to the aforementioned field equations is challenging due to the highly nonlinear nature of the underlying differential equations and the intense interactions among the metric functions. In [16], some of the authors introduced a novel technique for producing perfect fluid solutions to the Einstein equations, initiating from a selected seed solution. This approach involves adding a new parameter to the metric functions of the seed solution, thereby creating a new solution with physical characteristics distinct from those of the original seed solutions.

To construct the field equations for slightly deformed static objects for interior structure, we can use the approximate line element in [17] as a guide. Following this procedure, an appropriate interior line element can be expressed as

$$\begin{aligned} ds^2 = e^{2\nu}(1 + q\alpha) dt^2 - (1 + qc + qb)\frac{dr^2}{1 - \frac{2\tilde{m}}{r}} - \\ (1 + qa + qb) r^2 d\theta^2 - (1 - qa) r^2 \sin^2\theta d\phi^2 , \end{aligned} \tag{5}$$

where the functions $\nu = \nu(r), a = a(r), c = c(r), \tilde{m} = \tilde{m}(r), b = (r,\theta)$.

The corresponding linearized Einstein equations can be represented as

$$\overset{(0)}{G}{}_{\mu}^{\nu}+q\overset{(q)}{G}{}_{\mu}^{\nu}=8\pi\left(\overset{(0)}{T}{}_{\mu}^{\nu}+q\overset{(q)}{T}{}_{\mu}^{\nu}\right), \tag{6}$$

where the (0)−terms correspond to the limiting case of spherical symmetry. As for the energy-momentum tensor, we assume that density and pressure can also be linearized as

$$p(r)=p_0(r)+qp_1(r),\quad \rho(r)=\rho_0(r)+q\rho_1(r), \tag{7}$$

in accordance with the conservation law conditions (5). Here, $p_0(r)$ and $\rho_0(r)$ are the pressure and density of the background spherically symmetric solution, respectively. The significance of presenting the approximate line elements as mentioned above lies in the fact that it facilitates a simpler process for aligning the interior and exterior metrics.

Matching conditions

For the exterior gravitational field of slightly deformed objects, the parameter q in the exterior metric can be considered as small, then we can linearize the line element as [17,18]

$$ds^2=\left(1-\frac{2m}{r}\right)\left[1+q\ln\left(1-\frac{2m}{r}\right)\right]dt^2-r^2\left[1-q\ln\left(1-\frac{2m}{r}\right)\right]\sin^2\theta d\phi^2-$$
$$-\left[1+q\ln\left(1-\frac{2m}{r}\right)-2q\ln\left(1-\frac{2m}{r}+\frac{m^2}{r^2}\sin^2\theta\right)\right]\cdot\left(\frac{dr^2}{1-\frac{2m}{r}}+r^2d\theta^2\right) \tag{8}$$

Furthermore, this line element represents a particular approximate solution to Einstein's equations in vacuum.

Specifically, by examining the boundary conditions at the matching surface $r=r_\Sigma$ and comparing the aforementioned interior metric (6) with the approximate exterior q-metric (9) at the first order in q, we can derive the conditions for matching as

$$a(r_\Sigma)=-\frac{2m}{r_\Sigma-m},\ c(r_\Sigma)=-\frac{2m}{(r_\Sigma-m)^2},\ b(r_\Sigma)=-\frac{4m}{r_\Sigma-m}+\alpha$$
$$\nu(r_\Sigma)=\frac{1}{2}\ln\left(1-\frac{2m}{r_\Sigma}\right),\ m(r_\Sigma)=m. \tag{9}$$

we can impose the physically meaningful condition that the total pressure vanishes at the matching surface, i.e.

$$\tilde{m}(\Sigma)=m,\ p(\Sigma)=0. \tag{10}$$

From the point of view of a numerical integration, the above matching conditions can be used as boundary values for the integration of the corresponding differential equations. It's important to note that achieving the necessary alignment is done by setting the spatial coordinate to $r=r_\Sigma$ alone. Yet, as highlighted in the preceding section, this does not imply that the matching surface forms a sphere. In reality, the form of the matching surface is shaped by the conditions $t=const.$ and $r=r_\Sigma$, which, as stated by Eq. (11), specify a surface that varies explicitly with θ. Consequently, the coordinate *r* does not serve as a radial coordinate.

Discussion

Solving the field equations with the Equation of state

To find the interior metrics to describe the internal gravitational field of white dwarfs and neutron stars, we should integrate the equation of state (EoSs) together with the field equations. The explicit form of the corresponding field equations is given in [17]. This set of equations can be integrated immediately once the Equation of state is known. In particular, for a constant density of mass distribution i.e., ρ=*const.*, we obtain the interior Schwarzschild metric, which is the zeroth-order solution of Einstein's field equation.

We can provide an analyze to show the possibility of finding physical solutions for the field equations with the additional quadrupole parameter q for the given EoS.

An interesting class of fluids are the barotropic fluids, which obey the EoS [24]. One of the simplest cases is represented by the barotropic relation

$$p = \omega\rho \tag{11}$$

where *w* is the barotropic constant factor.

The behavior of the resulting total density is represented in Fig.1.

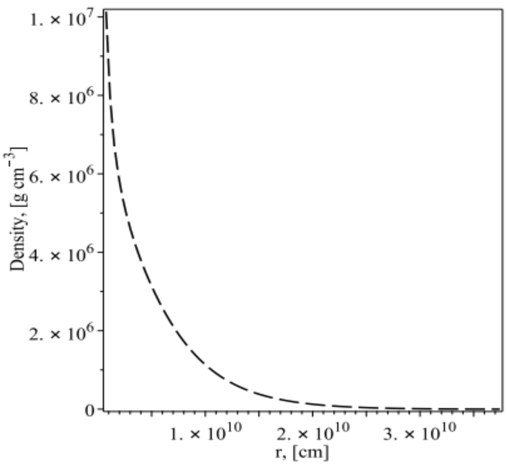

Fig. 1 - Behavior of the total density in terms of spatial coordinate r in geometrized units with G = c = 1, where $\rho_0 = 10^6\, гр/см^3$

To proceed with the numerical integration of the field equations in this case, the free parameters as

$$\rho_0 = \frac{3m}{4\pi R^3}, R = 1, m = 0.435, q = \frac{1}{100} \tag{12}$$

An interesting relation on the density and pressure functions of barotropic EoS presented as following

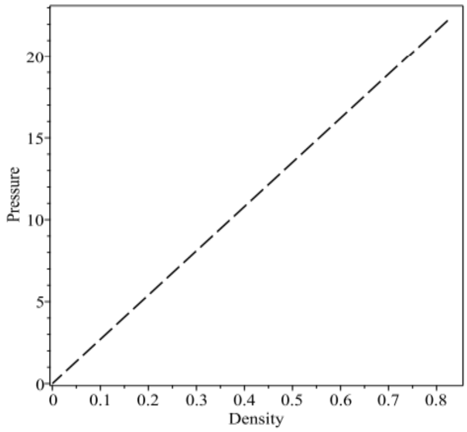

Fig. 2 - Barotropic behavior of pressure P versus density ρ in geometrized units, where ω = 27

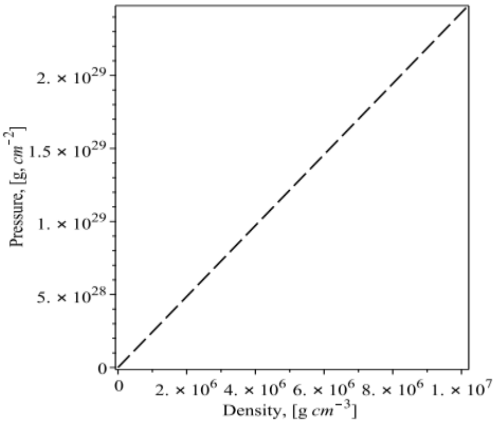

Fig. 3. - Barotropic behavior of pressure versus density P(ρ) in cgs units, where ω = 27

Accordingly, we will compare the physical properties of solutions, as expressed at the level of the EoS, with those of realistic compact objects. Consider, for instance, a neutron stars whose interior is described by the Chandrasekhar EoS in parametric form and geometrized units [16,25]:

$$\varepsilon = \frac{e_0}{8}\left[\left(2y(x)^3 + y(x)\right)\sqrt{1+y(x)^2} - \ln\left(y(x)+\sqrt{1+y(x)^2}\right)\right]$$
$$p = \frac{e_0}{24}\left[\left(2y(x)^3 - 3y(x)\right)\sqrt{1+y(x)^2} + 3\ln\left(y(x)+\sqrt{1+y(x)^2}\right)\right] \qquad (13)$$

where $e_0 = m_n^4 c^5 / \pi^2$ is the energy density.

Figure 4 illustrates the behavior of the equation of state (EoS) for pure degenerate neutron gas, which we adopt here for simplicity. We observe behavior comparable to that obtained from the numerical solutions presented above.

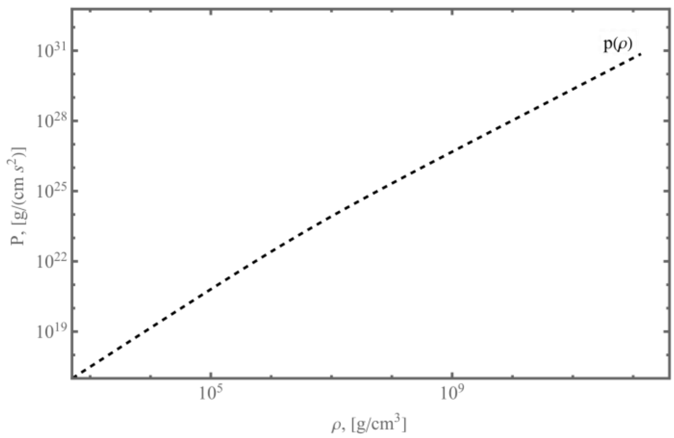

Fig. 4. Behavior of pressure versus density for the Chandrasekhar EoS.

In our analysis, we proposed using a different method that involves finding an effective EoS from the above parametric EoS and plotting the result as an effective EoS of pressure versus density. It was shown that in both cases, the effective EoS can be approximated by a polytropic equation of state, and the appropriate polytropic parameters have been identified. We found that the internal properties of these compact objects can be effectively represented by a polytropic EoS, which essentially mirrors the behavior observed in the numerical solutions.

The Salpeter EoS is particularly useful for modeling the interior of white dwarfs, where temperatures are relatively low, and electron degeneracy pressure supports the star against gravitational collapse[26,27]. It incorporates the effects of electron degeneracy without considering strong interactions between particles, making it ideal for white dwarfs that do not reach the densities where such interactions become significant. The Feynman-Metropolis-Teller EoS provides insights into the arrangement of atomic nuclei at extreme densities, promising to refine our understanding of the internal structure of compact objects[28]. We obtain numerical solutions that satisfy the matching conditions for the metric functions and the energy conditions for both EoSs.

Conclusion

In this study, have focused the structure of dense objects in space, like white dwarfs and neutron stars, using Einstein's theory of gravity, especially when they are not perfectly spherical. We pay close attention to a property called the quadrupole moment, which helps us better describe their shape and gravity.

To analyze these objects, we constricted field equations that include the quadrupole moment. A significant part of our research involves selecting the right equations of state (EoS), which is important for studying the inner pressures and densities of white dwarfs and neutron stars.

One of the main challenges we addressed is making sure that our solutions for the inside of these objects match well with the solutions for the space outside them. This process, known as matching conditions, is crucial for our solutions to be realistic and reflect what we can observe about these objects in space.

Through computational methods, we solved equations to find out more about the internal structure of compact objects and the results checked with the well-known EoS and verified that these particular solutions can be applied to describe the exterior and interior gravitational field of compact objects. The provide analyze shows that it is possible to find solutions for the internal makeup of white dwarfs and neutron stars using the Salpeter EoS and the Feynman-Metropolis-Teller EoS. Both EoS are foundational for astrophysical models of compact objects, enabling scientists to predict and understand the complex behavior of matter under conditions that are impossible to replicate on Earth. They are instrumental in studying the structure, evolution, and dynamics of white dwarfs and neutron stars, offering insights into the fundamental properties of matter in the universe.

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