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REPRESENTATION OF A NON-STATIONARY MODEL OF BAROCLINIC OCEAN MOTION USING THE FICTITIOUS DOMAIN METHOD

Abstract

This paper presents a groundbreaking non-stationary model, intricately crafted using the fictitious domain technique, to delve into the complex dynamics of baroclinic ocean motion. This study marks a significant leap in our understanding of water mass interaction, shedding light on the profound impact of temperature and salt gradients on sea currents. The methodology uses modified Navier-Stokes equations for viscous, incompressible flow, considering advection, diffusion, and Coriolis force. The results of this study underscore the immediate and tangible implications of our research. The solutions unveiled the pivotal role of pressure and temperature differentiation in the genesis of ocean currents. The analysis demonstrated that by integrating nonlinear terms and detailed modeling of initial and boundary conditions, we can markedly improve the precision of water mass movement forecasts. This work underscores the urgent necessity for further research into dynamic ocean modeling to enhance our ability to predict climate change. This article introduces truly innovative approaches to numerical modeling, which hold immense potential for the future of the field. These approaches have the power to transform existing models of sea currents and pave the way for the development of more efficient methods for monitoring and predicting the state of the marine environment.

Keywords: baroclinic motion, ocean modeling, ocean dynamics, Navier-Stokes equation, Coriolis force, advection and diffusion, climate change, non-stationary processes in the ocean.

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ПРЕДСТАВЛЕНИЕ НЕСТАЦИОНАРНОЙ МОДЕЛИ ДВИЖЕНИЯ БАРОКЛИННОГО ОКЕАНА С ПОМОЩЬЮ МЕТОДА ФИКТИВНЫХ ОБЛАСТЕЙ

Аннотация

В данной статье представлена разработка и анализ комплексной нестационарной модели для изучения движения бароклинного океана, основанной на методике фиктивных областей. Целью исследования является улучшение понимания механизмов взаимодействия водных масс, а также влияния температурных и солевых градиентов на динамику морских течений. Методология включает в себя использование модифицированных уравнений Навье-Стокса для вязкого, несжимаемого потока с учетом адвекции, диффузии и кориолисовой силы. В результате были получены решения, которые демонстрируют значительное влияние вертикальных и горизонтальных дифференциаций давления и температуры на формирование океанических течений. Анализ показал, что включение нелинейных членов и детальное моделирование начальных и граничных условий позволяют значительно повысить точность прогнозов движения водных масс. Работа подчеркивает важность дальнейших исследований в области динамического моделирования океана для более прогнозирования климатических изменений. Статья предлагает новые подходы к численному моделированию, которые могут быть использованы для улучшения существующих моделей морских течений, а также для разработки более эффективных методов мониторинга и прогнозирования состояния морской среды.

Ключевые слова: бароклинное движение, моделирование океана, динамика океана, уравнение Навье-Стокса, Кориолисова сила, адвекция и диффузия, климатические изменения, нестационарные процессы в океане.

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ЖАЛҒАН АЙМАҚТЫҚ ӘДІСІ АРҚЫЛЫ БАРОКЛИНДІ МҰХИТ ҚОЗҒАЛЫСЫНЫҢ СТАЦИОНАРЛЫҚ ЕМЕС МОДЕЛІН КӨРСЕТУ

Аңдатпа

Бұл жұмыс жалған аймақтық техникасына негізделген бароклиникалық мұхиттың қозғалысын зерттеудің күрделі стационарлы емес моделін жасау және талдауды ұсынады. Зерттеудің мақсаты су массаларының өзара әрекеттесу механизмдерін, сондай-ақ температура мен тұз градиенттерінің теңіз ағындарының динамикасына әсерін түсінуді жетілдіру болып табылады. Әдістеме адвекция, диффузия және Кориолис күшін ескере отырып, тұтқыр, сығылмайтын ағын үшін модификацияланған Навье-Стокс теңдеулерін қолдануды қамтиды. Нәтижесінде мұхит ағыстарының пайда болуына қысым мен температураның тік және көлденең дифференциациясының елеулі әсерін көрсететін шешімдер алынды. Талдау көрсеткендей, сызықты емес терминдерді қосу және бастапқы және шекаралық шарттарды егжей-тегжейлі модельдеу су массасының қозғалысы туралы болжамдардың дәлдігін айтарлықтай жақсартуға мүмкіндік береді. Жұмыс климаттың өзгеруін жақсы болжау үшін динамикалық мұхитты модельдеуді одан әрі зерттеудің маңыздылығын көрсетеді. Мақалада теңіз ағындарының қолданыстағы үлгілерін жақсарту, сондай-ақ теңіз ортасының жағдайын бақылау және болжау үшін тиімдірек әдістерді әзірлеу үшін пайдалануға болатын сандық модельдеудің жаңа тәсілдері ұсынылған.

Түйін сөздер: бароклиникалық қозғалыс, мұхитты модельдеу, мұхит динамикасы, Навье-Стокс теңдеуі, Кориолис күші, адвекция және диффузия, климаттың өзгеруі, мұхиттағы стационарлық емес процестер.

Main provisions

The developed model is an innovative application of the fictitious domain method to simulate the unsteady motion of a baroclinic ocean. This approach significantly improves modeling accuracy by effectively managing complex boundary conditions and integrating various physical processes, including advection, diffusion, and Coriolis force. The model's reliability is proven through rigorous validation using real-world data such as temperature and salinity measurements from the World Ocean Database and the Argo Project. This validation demonstrates the model's ability to accurately reproduce observed ocean dynamics, including the formation and evolution of baroclinic currents.

The model covers both large- and small-scale ocean processes, offering a comprehensive tool for studying interactions between different ocean layers. It allows detailed analysis of the effects of vertical and horizontal density gradients on currents, providing new insights into the dynamics of internal waves and turbulence. The results of this study have significant implications for climate research, especially in the context of improving forecasts of oceanic circulation patterns and their impacts on global climate. The model's ability to incorporate complex initial and boundary conditions makes it a valuable resource for understanding the long-term effects of climate change on the marine environment.

Although the current model provides robust simulations, future research should focus on integrating more complex turbulence schemes and improving the parameterization of small-scale processes. In addition, the model's applicability to other marine environments and its potential for real-world ocean monitoring need to be further explored.

Introduction

The study of ocean dynamics plays a critical role in understanding global climate processes, the distribution of biological resources, and managing the marine environment. Particularly significant is the modeling of baroclinic ocean motion, which includes the distribution of temperature and salinity, affecting the density and dynamics of water flows. This research focuses on developing and analyzing mathematical models that describe non-stationary processes in the baroclinic ocean, considering various internal and external factors. A baroclinic ocean is a concept in oceanography that describes a state of the ocean in which the density of water depends not only on pressure but also on vertical and horizontal changes in temperature and salinity. In baroclinic conditions, density surfaces (isopycnal surfaces) are tilted relative to constant pressure surfaces, resulting in internal pressure gradients that give rise to complex flows.

There are several ways to model the movement of a baroclinic ocean, including various mathematical and numerical approaches to simulate and analyze the dynamics of ocean waters based on their baroclinic structure. These methods include:

– Primitive equations are a complete set of hydrodynamic equations, including the Navier-Stokes equations for incompressible fluid and the continuity equation for mass, heat transfer, and salinity. Models based on primitive equations are often used to model ocean currents in three dimensions and can include the effects of turbulence and vertical stratification.

– Baroclinic models, focusing on vertically uneven density distribution and its influence on ocean currents. Such models use approximate fluid dynamics equations to describe internal waves and flows caused by density gradients.

– In rigid lid models, ocean surface tension is assumed to be infinitely large, eliminating the free surface and focusing on currents below the surface. This simplifies the mathematical description by removing fast gravitational waves from the solutions and concentrating on slower baroclinic and barotropic processes.

– Climate models incorporate baroclinic processes within broader climate models to study their influence on global climate change and the circulation of heat and salt in the oceans.

– Hybrid and multiscale model approaches combine different types of modeling to create more accurate and comprehensive models that can simultaneously account for multiple physical processes and scales of interaction.

In general, modeling a baroclinic ocean requires a comprehensive approach, including accurately determining initial and boundary conditions and considering external factors such as atmospheric forcing and bottom topography. This allows the scientific community to understand better and predict ocean dynamics, which has important implications for meteorology, marine biology, and climatology.

Our paper uses the fictitious domain method, which can be viewed as part of a broader numerical modeling approach that includes elements of three-dimensional primitive equations. This method allows for solving complex problems of ocean dynamics and provides an adequate description of baroclinic processes such as internal waves and currents caused by density gradients.

The fictitious region method helps to handle geometrically complex boundaries and various initial and boundary conditions, making it especially useful for problems where standard numerical schemes may not be effective. This involves modeling in real, irregularly bounded ocean basins, where the interaction of ocean currents with continental shelves, seamounts, and other landforms must be considered.

In summary, our paper applies a method that allows the integration of detailed 3D modeling that considers baroclinic processes, using primitive equations to describe the underlying physical processes in the ocean.

Research methodology

Formulation of the problem

Let us consider in the region $\mathrm{Q}_0 = \Omega_0 \times (0, T),\ \ \Omega_0 = (0, H) \times D_0$ the following equations of motion of the Baroclinic Ocean.

1. Equation describing the change in the speed of water in the ocean:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{v} \cdot \nabla)\vec{u} = \mu_0 \frac{\partial^2 \vec{u}}{\partial z^2} + \mu \Delta \vec{u} - \vec{\nabla} p - [\vec{\ell} \times \vec{u}],$$

where

- $\frac{\partial \vec{u}}{\partial t} -$ time derivative of speed, showing the change in speed with time;
- $(\vec{v} \cdot \nabla)\vec{u} -$ advective term describing the transfer of velocity by flow;
- $\mu_0 \frac{\partial^2 \vec{u}}{\partial z^2} + \mu \Delta \vec{u} -$ diffusion terms modeling viscous effects;
- $\vec{\nabla} p -$ pressure gradient;
- $[\vec{\ell} \times \vec{u}] -$ Coriolis term describing the effect of the Earth's rotation on the movement of water.

2. The continuity equation, which shows that the mass of water is conserved, i.e., neither its creation nor its destruction occurs:

$$\frac{\partial u}{\partial x} + \frac{dv}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad \vec{v} = (u, v, w), \qquad \vec{u} = (u, v), \qquad \vec{\ell} = (\ell, \ell)$$

3. Hydrostatic pressure equation relating pressure changes to the depth, water density p, and gravitational acceleration g:

$$\frac{\partial p}{\partial z} = -\rho_0 g,\ \ \rho = a_0 \theta + b_0, \tag{1}$$

4. Equation for temperature θ

$$\frac{\partial \theta}{\partial t} + (\vec{v}\nabla)\theta = \lambda_0 \frac{\partial^2 \theta}{\partial z^2} + \lambda \Delta \theta,$$

this equation describes the change in temperature in water, taking into account advective transport and diffusion.

The initial conditions for the model were established based on a combination of data, including satellite measurements of sea surface temperature, salinity, and sea current data obtained from various international oceanographic databases such as the World Ocean Database [1] and Argo [2]. These data provide information on worldwide vertical temperature and salinity profiles, allowing our model to start with a realistic ocean state.

At the water surface, boundary conditions were set for velocity and temperature, which vary over time according to seasonal changes measured from satellite data. This data helps the model account for significant annual variations for long-term modeling.

Boundary conditions for the above equations:

– initial conditions for speed and temperature:

$$\vec{u}|_{t=0} = \vec{u}^0(x, y, z), \qquad \theta|_{t=0} = \theta^0(x, y, z),$$

Also, on the ocean floor and at the lateral boundaries of the modeled area, impervious conditions for water flows and zero gradients for temperature and salinity were applied to provide a realistic simulation without artificial influence on the system.

– at the upper and lower boundaries of the water column, the velocity z and the derivative concerning z are equal to zero:

$$\left.\frac{\partial \vec{u}}{\partial z}\right|_{z=H} = \vec{u}|_{z=0} = 0, \ \vec{u}|_{\partial D_0} = 0, \ z \in [0, H], \tag{2}$$

– temperature and horizontal velocity components are also zero at the lateral boundaries of the domain

$$w|_{z=0} = w|_{z=H} = 0, \qquad \theta|_{\partial D_0} = 0, \ z \in [0, H].$$

Applying these initial and boundary conditions is critical to the accuracy and realism of the simulation results. Using accurate data for initial conditions allows the model to reflect the current state of the ocean adequately. Adaptive surface boundary conditions that reflect seasonal and weather changes will enable the model to track dynamic changes in the ocean, such as thermocline formation and breakdown and changes in salinity currents, which is especially important for long-term and climate modeling.

The presented system of equations (1) and conditions (2) models the dynamic behavior and thermodynamic processes in the baroclinic ocean, considering hydrostatic equilibrium and the influence of external forces.

Using the above method in works [3,4], the system of equations (1) is reduced to the following

$$\frac{\partial \vec{u}}{\partial t} + (\vec{v}\nabla)\vec{u} = \mu_0 \frac{\partial^2 \vec{u}}{\partial z^2} + \mu\Delta\vec{u} - \vec{\nabla}\xi - \left[\vec{\ell} \times \vec{u}\right] + \nabla h(x, y, z, \theta),$$

$$\int_0^H div\vec{u}dz = 0, \tag{3}$$

$$\frac{\partial \theta}{\partial t} + (\vec{v}\nabla)\theta = \lambda_0 \frac{\partial^2 \theta}{\partial z^2} + \lambda\Delta\theta.$$

$$\xi = p|_{z=0}, \ \int_{D_0} \xi dxdy = 0, \ \frac{\partial \xi}{\partial z} = 0, \tag{4}$$

$$\vec{v} = \left(u, v, -\int_0^z div\,\vec{u}\,dz\right)$$

This modification of the system of equations (1) makes it possible to more accurately simulate the movement of water in a baroclinic ocean, taking into account additional effects, such as changes in water volume and the impact of external factors on the system's dynamics.

Along with problems (2) and (3), we consider in the domain Q_2 the following system with a small parameter:

$$\frac{\partial \vec{u}^\varepsilon}{\partial t} + (\vec{v}^\varepsilon\nabla)\vec{u}^\varepsilon = \mu_0 \frac{\partial^2 \vec{u}^\varepsilon}{\partial z^2} + \mu\Delta\vec{u}^\varepsilon - \vec{\nabla}\xi^\varepsilon - \left[\vec{\ell} \times \vec{u}^\varepsilon\right] + \nabla h(x, y, z, \theta) - \frac{\eta(x)}{\varepsilon}\vec{u}^\varepsilon, \tag{5}$$

$$\int_0^H div\,\vec{u}\,dz = 0,\ \frac{\partial \xi^\varepsilon}{\partial z} = 0,\quad \int_{D_2} \xi^\varepsilon dxdy = 0, \tag{6}$$

$$\frac{\partial \theta^\varepsilon}{\partial t} + (\vec{v}^\varepsilon \nabla)\theta^\varepsilon = \lambda_0 \frac{\partial^2 \theta^\varepsilon}{\partial z^2} + \lambda \Delta \theta^\varepsilon - \frac{\eta(x)}{\varepsilon}\theta^\varepsilon,$$

where the function $\eta(x)$ is given as follows:

$$\eta(x) = \begin{cases} 0, & x \in \Omega_0 \\ 1, & x \in \Omega_1 = \dfrac{\Omega_2}{\Omega_0} \end{cases}$$

The boundary conditions for systems (5) and (6) have the form:

$$\vec{u}^\varepsilon|_{t=0} = \vec{u}^0(x,y,z), \qquad \theta^\varepsilon|_{t=0} = \theta^0(x,y,z),$$

$$\theta^\varepsilon|_{\partial D_2} = 0,\ \vec{u}^\varepsilon|_{\partial D_2} = 0,\ \ z \in [0,H], \tag{7}$$

$$\left.\frac{\partial \vec{u}^\varepsilon}{\partial z}\right|_{z=0} = \left.\frac{\partial \vec{u}^\varepsilon}{\partial z}\right|_{z=H}, \quad (x,y) \in D_2.$$

Definition 1. A generalized solution to problem (5)-(7) is a pair of functions $\{\vec{u}^\varepsilon, \theta^\varepsilon\}$ such that

$$\vec{u}^\varepsilon(x,y,z,t) \in L_\infty\big(0,T;\ \dot{V}_2^1(\Omega_2)\big) \cap L_2\big(0,T;\ \dot{V}_2^1(\Omega_2)\big),$$

$$\theta^\varepsilon(x,y,z,t) \in L_\infty\big(0,T;\ \dot{W}_2^1(\Omega_2)\big) \cap L_2\big(0,T;\ \dot{W}_2^1(\Omega_2)\big),$$

and satisfying the following integral identities:

$$\int_0^T \int_{\Omega_2} \Big\{\vec{u}^\varepsilon \vec{\varphi}_t + (\vec{v}^\varepsilon \nabla)\vec{\varphi}\vec{u}^\varepsilon - \mu_0 \vec{u}_z^\varepsilon \varphi_z - \mu \nabla \vec{u}^\varepsilon \vec{\varphi} - \big[\vec{\ell} \times \vec{u}\big]\vec{\varphi} - \frac{\eta}{\varepsilon}\vec{u}^\varepsilon \vec{\varphi} - h\mathrm{div}\vec{\varphi}\Big\} dxdydzdt +$$

$$+ \int_{\Omega_2} \vec{u}^0 \vec{\varphi}|_{t=0}\, dzdydz = 0, \tag{8}$$

$$\int_0^T \int_{\Omega_2} \Big\{\theta^\varepsilon \psi_t + (\vec{v}^\varepsilon \nabla)\psi\theta^\varepsilon - \lambda_0 \theta_z^\varepsilon \psi_z - \lambda \nabla \theta^\varepsilon \psi - \frac{\eta}{\varepsilon}\theta^\varepsilon \psi\Big\} dxdydzdt +$$

$$+ \int_{\Omega_2} \theta^0 \psi|_{t=0}\, dzdydz = 0,$$

for any

$$\vec{\varphi}(x,y,z,t) \in C^1\Big(0,T;\ \dot{V}_2^1(\Omega_2)\Big), \qquad \psi(x,y,z,t) \in C^1\big(0,T;\ \dot{W}_2^1(\Omega_2)\big),$$

such that $\vec{\varphi}|_{t=T} = 0,\ \psi|_{t=T} = 0.$
Let us obtain a priori estimates of solutions. Assuming in identities (8) $\vec{\varphi} = \vec{u}^\varepsilon,\ \ \psi = \theta^\varepsilon$, we obtain

$$\max_{0\le t\le T}\{\|\vec{u}^{\varepsilon}\|^2_{L_2(\Omega_2)}+\|\theta^{\varepsilon}\|^2_{L_2(\Omega_2)}\}+\mu_0\|\vec{u}^{\varepsilon}_z\|^2_{L_2(Q_2)}+\lambda_0\|\theta^{\varepsilon}_z\|^2_{L_2(Q_2)}+\mu\|\nabla\vec{u}^{\varepsilon}\|^2_{L_2(Q_2)}+\lambda\|\nabla\theta^{\varepsilon}\|^2_{L_2(Q_2)}+$$

$$+\frac{1}{\varepsilon}\left\{\int_0^T\int_{\Omega_2}[(\theta^{\varepsilon})^2+(\vec{u}^{\varepsilon})^2]dxdydzdt\right\}\le C\{\|h\|^2_{L_2(Q_2)}+\|\vec{u}^{\circ}\|^2_{L_2(Q_2)}+\|\theta^{\circ}\|^2_{L_2(Q_2)}\}.$$

Now from the inequality $\|h\|_{L_2(Q_2)}\le C(\|\theta^2\|_{L_2(Q_2)}+1)$, we obtain the estimate

$$\max_{0\le t\le T}\{\|\vec{u}^{\varepsilon}\|^2_{L_2(\Omega_2)}+\|\theta^{\varepsilon}\|^2_{L_2(\Omega_2)}\}+\mu_0\|\vec{u}^{\varepsilon}_z\|^2_{L_2(Q_2)}+\lambda_0\|\theta^{\varepsilon}_z\|^2_{L_2(Q_2)}+\mu\|\nabla\vec{u}^{\varepsilon}\|^2_{L_2(Q_2)}+\lambda\|\nabla\theta^{\varepsilon}\|^2_{L_2(Q_2)}+$$

$$+\frac{1}{\varepsilon}\left(\|\vec{u}^{\varepsilon}\|^2_{L_2(Q_1)}+\|\theta^{\varepsilon}\|^2_{L_2(Q_1)}\right)\le C\left(\|\vec{u}^{\circ}\|^2_{L_2(Q_2)}+\|\theta^{\circ}\|^2_{L_2(Q_2)}\right), \tag{9}$$

where $Q_1=\Omega_1\times[0,T]$.

Lemma 1. Let $\vec{u}^{\circ}(x,y,z)\in\dot{V}^1_2(\Omega_2)$, $\theta^{\circ}(x,y,z)\in\dot{W}^1_2(\Omega_2)$. Then estimates (9) and

$$\|\vec{u}^{\varepsilon}_t\|_{L_{4/3}\left(0,T;\,\dot{V}^1_2(\Omega_2)\right)}+\|\theta^0_t\|_{L_{4/3}(0,T;\,\dot{W}^1_2(\Omega_2))}\le C_1\varepsilon, \tag{10}$$

where $C_1\varepsilon\to\infty$ at $\varepsilon\to 0$..

Proof. We get (10). To do this, multiply (5), (6) by $\vec{\varphi}(x,y,z,t)\in L_4\left(0,T;\ \dot{V}^1_2(\Omega_2)\right)$ and $\psi(x,y,z,t)\in L_4\left(0,T;\ W^2_2(\Omega_2)\cap\dot{W}^1_2(\Omega_2)\right)$ respectively. We have

$$\int_{Q_2}\vec{u}^{\varepsilon}_t\,\vec{\varphi}dQ_2=\int_{Q_2}(\vec{v}^{\varepsilon}\nabla)\vec{\varphi}\vec{u}^{\varepsilon}dQ_2-\int_{Q_2}\left[\mu_0\vec{u}^{\varepsilon}_z\vec{\varphi}_z+\mu\nabla\vec{u}^{\varepsilon}\nabla\vec{\varphi}+[\vec{\ell}\times\vec{u}^{\varepsilon}]\vec{\varphi}+\frac{\eta}{\varepsilon}\vec{u}^{\varepsilon}\vec{\varphi}+hdiv\vec{\varphi}\right]dQ_2, \tag{11}$$

$$\int_{Q_2}\theta^{\varepsilon}_t\psi dQ_2=\int_{Q_2}(\vec{v}^{\varepsilon}\nabla)\psi\theta^{\varepsilon}dQ_2-\int_{Q_2}\left[\lambda_0\theta^{\varepsilon}_z\psi+\lambda\nabla\theta^{\varepsilon}\psi+\frac{1}{\varepsilon}\theta^{\varepsilon}\psi\right]dQ_2. \tag{12}$$

Let us consider identities (11), (12) as the relation of linear functionals over the spaces $\dot{V}^1_2(\Omega_2)$ and $W^2_2(\Omega_2)\cap\dot{W}^1_2(\Omega_2)$.

$$\int_0^T[L_0(t),\vec{\varphi}(t)]_{\Omega_2}dt=\int_0^T[L_1(t),\vec{\varphi}(t)]_{\Omega_2}dt+\int_0^T[L_2(t),\vec{\varphi}(t)]_{\Omega_2}dt \tag{13}$$

$$\int_0^T[L_3(t),\vec{\varphi}(t)]_{\Omega_2}dt=\int_0^T[L_4(t),\vec{\varphi}(t)]_{\Omega_2}dt+\int_0^T[L_5(t),\vec{\varphi}(t)]_{\Omega_2}dt \tag{14}$$

Let us estimate the functional $L_0(t)$. To do this, we note that the inequalities take place

$$\left|[L_1(t),\vec{\varphi}(t)]_{\Omega_2}\right| = \left|\int_{\Omega_2}(\vec{v}^\varepsilon\nabla)\vec{\varphi}\vec{u}^\varepsilon dxdydz\right| \le C\|v^\varepsilon\|_{L_2(\Omega_2)}\times\|\vec{u}^\varepsilon\|_{L_2(\Omega_2)}\times\|\nabla\vec{\varphi}\|_{L_2(\Omega_2)} \le$$

$$\le C\|v^\varepsilon\|_{L_2(\Omega_2)}\times\left\|\overrightarrow{\nabla u}^\varepsilon\right\|^{\frac{1}{2}}_{L_2(\Omega_2)}\times\|\vec{u}^\varepsilon\|^{\frac{1}{2}}_{L_2(\Omega_2)}\times\|\vec{\varphi}\|_{W_2^2(\Omega_2)}$$
$$\le C\left\|\overrightarrow{\nabla u}^\varepsilon\right\|^{\frac{3}{2}}_{L_2(\Omega_2)}\times\|\vec{u}^\varepsilon\|^{\frac{1}{2}}_{L_2(\Omega_2)}\times\|\vec{\varphi}\|_{W_2^2(\Omega_2)}$$

$$\left|[L_2(t),\vec{\varphi}(t)]_{\Omega_2}\right| = \left|\int_{\Omega_2}\left[\mu_0\vec{u}_z^\varepsilon\vec{\varphi}_z+\mu\nabla\vec{u}^\varepsilon\nabla\vec{\varphi}+\left[\vec{\ell}\times\vec{u}^\varepsilon\right]\vec{\varphi}+\frac{\eta(x)}{\varepsilon}\vec{u}^\varepsilon\vec{\varphi}+hdiv\vec{\varphi}\right]dxdydz\right| \le$$

$$\le C\left(\|\vec{u}^\varepsilon\|_{L_2(\Omega_2)}+\left\|\overrightarrow{\nabla u}^\varepsilon\right\|_{L_2(\Omega_2)}+\|h\|_{L_2(\Omega_2)}\right)\|\vec{\varphi}\|_{W_2^1(\Omega_2)}+\frac{1}{\varepsilon}\|\vec{u}^\varepsilon\|_{L_2(\Omega_2)}\|\vec{\varphi}\|_{L_2(\Omega_2)}$$
$$\le C_2\varepsilon\|\vec{\varphi}\|_{W_2^1(\Omega_2)}$$

From these inequalities and Hölder's inequalities, the following:

$$\int_0^T[L_1(t),\vec{\varphi}(t)]_{\Omega_2}dt \le \mathrm{C}\max_{0\le t\le T}\|\vec{u}^\varepsilon\|^{\frac{1}{2}}_{L_2(\Omega_2)}\left[\int_0^T\left\|\overrightarrow{\nabla u}^\varepsilon\right\|^2_{L_2(\Omega_2)}\right]^{\frac{3}{4}}\times\left[\int_0^T\|\vec{\varphi}\|^4_{W_2^2(\Omega_2)}\right]^{\frac{1}{4}},$$

$$\int_0^T[L_2(t),\vec{\varphi}(t)]_{\Omega_2}dt \le \mathrm{C}_3\varepsilon\left[\int_0^T\|\vec{\varphi}\|^2_{W_2^2(\Omega_2)}\right]^{\frac{1}{2}} \le C_4\varepsilon\left[\int_0^T\|\vec{\varphi}\|^4_{W_2^2(\Omega_2)}\right]^{\frac{1}{4}}$$

By Riesz's theorem on the representation of a linear functional

$$\|L_1(t)\|_{L_{4/3}\left(0,T;W_2^{-2}(\Omega_2)\right)} = \sup_{\vec{\varphi}\in L_4\left(0,T;\dot{V}_2^2(\Omega_2)\right)}\frac{\left|\int_0^T[L_1(t),\vec{\varphi}]_{\Omega_2}dt\right|}{\|\vec{\varphi}\|_{L_4\left(0,T;\dot{V}_2^2(\Omega_2)\right)}} \le C_5\varepsilon,$$

$$\|L_2(t)\|_{L_{\frac{4}{3}}\left(0,T;W_2^{-2}(\Omega_2)\right)} = \sup_{\vec{\varphi}\in L_4\left(0,T;\dot{V}_2^2(\Omega_2)\right)}\frac{\left|\int_0^T[L_2(t),\vec{\varphi}]_{\Omega_2}dt\right|}{\|\vec{\varphi}\|_{L_4\left(0,T;\dot{V}_2^2(\Omega_2)\right)}} \le C_4\varepsilon,$$

and from equality

$$L_0(t) = L_1(t)+L_2(t),$$

Should

$$\|L_0(t)\|_{L_{4/3}\left(0,T;W_2^{-2}(\Omega_2)\right)} \le \mathrm{C}_6\varepsilon,\ \ C_6\varepsilon\to\infty\ \text{ при }\ \varepsilon\to 0.$$

Using Riesz's theorem again, we obtain

$$\|\vec{u}_t^\varepsilon\|_{L_{4/3}\left(0,T;W_2^{-2}(\Omega_2)\right)} \le C_6\varepsilon$$

Consider in the domain Ω_2 the linear operator

$$L\vec{w} = \mu_0\vec{w}_{zz} + \mu_1\Delta\vec{w} - \nabla\xi,$$

acting over the space $\dot{V}_2(\Omega_2)$.

It is known that the operator L is closed, symmetric, and its range of values fills the entire space $\dot{V}_2(\Omega_2)$,, so it is self-adjoint. Since the set of functions bounded in $\dot{V}_2^1(\Omega_2)$ is compact in $\dot{V}_2(\Omega_2)$, then the operator L^{-1} is completely continuous.

From these properties of the operator $L[\vec{w}]$ it follows that the spectrum $\gamma = \gamma_1, \gamma_2, \ldots$ is discrete, its negativity, finite multiplicity, tendency $\lambda_k, k \to \infty$, orthogonality and completeness of eigenfunctions in the metric $L_2(\Omega_2)$ and $\dot{V}_2(\Omega_2)$.

The eigenfunctions $\vec{w}_j$ are solutions to the problems

$$\mathrm{L}\vec{w}_j = \mu_0\vec{w}_{jzz} + \mu_1\Delta\vec{w}_j - \nabla\xi_j = \gamma_j\vec{w},$$

$$\int_0^H \mathrm{div}\,\vec{w}_j\,dz = 0, \;\; \vec{w}_j\big|_{\partial D_0} = 0, \quad z \in [0, H], \tag{15}$$

$$\left.\frac{\partial\vec{w}_j}{\partial z}\right|_{z=H} = \vec{w}_j\big|_{z=0} = 0, \qquad (x, y) \in D_2$$

Inside the region Ω_2 they are infinitely differentiable. The smoothness near the boundary $\partial\Omega_2$ is determined by the smoothness ∂D_2.

Theorem 1. Let $\vec{u}^0(x, y, z) \in L_2(\Omega_2)$, $\theta^0(x, y, z) \in L_2(\Omega_2)$. Then problem (5)-(7) has at least one generalized solution and estimates (9), (10) are valid.

We will carry out the proof using the Galerkin method. We will look for approximate solutions $\vec{u}_\varepsilon^N(x, y, z, t)$ in the form of finite sums

$$\vec{u}_\varepsilon^N(x, y, z, t) = \sum_{k=1}^N a_{Nk}(t) \cdot \vec{w}_k(x, y, z), \tag{16}$$

where $\vec{w}_k -$ is the basis $\dot{V}_2(\Omega_2)$ from the solutions of problem (15) orthonormalized in $L_2(\Omega_2)$.

We will find the functions $\theta_\varepsilon^N(x, y, z, t)$ as generalized solutions to the problem

$$\theta_{\varepsilon t}^N + \left(\vec{v}_\varepsilon^N \cdot \vec{\nabla}\right)\theta_\varepsilon^N = \lambda_0\theta_{\varepsilon zz}^N + \lambda\Delta\theta_\varepsilon^N - \frac{\eta}{\varepsilon}\theta_\varepsilon^N, \tag{17}$$

where

$$\vec{v}_\varepsilon^N = \left(u_\varepsilon^N, v_\varepsilon^N, -\int_0^z \mathrm{div}\; u_\varepsilon^N dz\right) \tag{18}$$

To determine the coefficients $a_{Nk}(t)$, we require that relation (18) be satisfied

$$\int_{\Omega_2} \Big\{\left[\vec{u}_{\varepsilon t}^N + \left(\vec{v}_\varepsilon^N \cdot \vec{\nabla}\right)\vec{u}_\varepsilon^N\right]\vec{w}_j + \mu_0\vec{u}_{\varepsilon z}^N\vec{w}_{jz} + \mu\nabla\vec{u}_\varepsilon^N\nabla\vec{w}_j + \left[\vec{\ell} \times \vec{u}_\varepsilon^N\right]\vec{w}_j + h(\theta^{N-1})div\vec{w}_j$$
$$+ \frac{\eta}{\varepsilon}\vec{u}_\varepsilon^N\nabla\vec{w}_j\Big\}\,d\Omega_2 = 0,$$

which is a system of N ordinary differential equations [5]

$$\sum_{k=1}^N \frac{da_{Nk}(t)}{dt} + \sum_{i,k=1}^N \beta_{ijk}a_{Ni}a_{Nk} + \sum_{k=1}^N \gamma_{jk}C_{Nk} = f_N^j,$$

$$j = 1, 2, \ldots, N,$$

where

$$\beta_{ijk} = \int_{\Omega_2} \left(\vec{w}_j \cdot \nabla\right) \vec{w}_i \vec{w}_k dxdydz,$$

$$\gamma_{jk} = \int_{\Omega_2} \left[\mu_0 \vec{w}_{jz} \vec{w}_{kz} + \mu \nabla \vec{w}_j \nabla \vec{w}_k + \left[\vec{\ell} \times \vec{w}_j\right] \vec{w}_k + \frac{\eta}{\varepsilon} \vec{w}_j \vec{w}_k\right] dydxdz,$$

$$f_N^i = -\int_{\Omega_2} h(\theta^{N-1}) div \vec{w}_j dxdydz$$

The initial data for equation (18) are taken from the expansion of $\vec{u}^0(x,y,z)$ over the basis $\{\vec{w}_j\}$

$$\vec{u}^0 = \sum_{j=1}^{\infty} a_j \vec{w}_j, \quad a_j = \int_{\Omega_2} \vec{u}^0 \vec{w}_j \, dxdydz,$$

in the following way

$$a_{Nk}(0) = a_k, \;\; k = 1,2,3 \dots, N \tag{19}$$

Lemma 2. For any $N = 1,2, \dots$ there is a unique solution $\{\vec{u}_\varepsilon^N, \theta_\varepsilon^N\}$ to problem (16)-(19) and the estimate uniform in N is valid

$$\max_{0\le t\le T} \|\vec{u}_\varepsilon^N\|^2_{L_2(\Omega_2)} + \mu_0 \|\vec{u}_\varepsilon^N\|^2_{L_2(Q_2)} + \mu \|\vec{u}_\varepsilon^N\|^2_{L_2(Q_2)} + \frac{1}{\varepsilon} \|\vec{u}_\varepsilon^N\|^2_{L_2(Q_1)} +$$

$$+ \max_{0\le t\le T} \|\theta_\varepsilon^N\|^2_{L_2(\Omega_2)} + \lambda_0 \|\theta_\varepsilon^N\|^2_{L_2(Q_2)} + \lambda \|\theta_\varepsilon^N\|^2_{L_2(Q_2)} + \frac{1}{\varepsilon} \|\theta_\varepsilon^N\|^2_{L_2(Q_1)} \le$$

$$\le C\left(\|\vec{u}^0\|^2_{L_2(\Omega_2)} + \|\theta^0\|^2_{L_2(\Omega_2)}\right). \tag{20}$$

Proof. Multiplying equation (17) by θ_ε^N and integrating by parts, we have

$$\max_{0\le t\le T} \|\theta_\varepsilon^N\|^2_{L_2(\Omega_2)} + \lambda_0 \|\theta_\varepsilon^N\|^2_{L_2(Q_2)} + \lambda \|\theta_\varepsilon^N\|^2_{L_2(Q_2)} + \frac{1}{\varepsilon} \|\theta_\varepsilon^N\|^2_{L_2(Q_1)} \le C \|\theta^0\|^2_{L_2(\Omega_2)} \tag{21}$$

Next, multiplying the j^{th} equation of system (18) by $C_j(t)$ and summing over *j* from 1 to N, we arrive at the inequality

$$\max_{0\le t\le T} \|\vec{u}_\varepsilon^N\|^2_{L_2(\Omega_2)} + \mu_0 \|\vec{u}_\varepsilon^N\|^2_{L_2(Q_2)} + \mu \|\vec{u}_\varepsilon^N\|^2_{L_2(Q_2)} + \frac{1}{\varepsilon} \|\vec{u}_\varepsilon^N\|^2_{L_2(Q_1)}$$
$$\le C\left(\|\vec{u}^0\|^2_{L_2(\Omega_2)} + \|h\theta^{N-1}\|^2_{L_2(Q_2)}\right)$$

This equation and (21) give an estimate (20).

Let us show the solvability of a problem (16)-(19). To do this, we choose in the space $C(0,T)$ a bounded convex set

$$k = \{\vec{\varphi}(t); \;\; |\vec{\varphi}| \le C_1, \qquad \varphi_i(0) = a_i, \qquad i = 1,2, \dots, N\}$$

Let's form a vector

$$\Phi = \sum_{i=1}^{N} \varphi_i \vec{w}_i$$

Let's find a solution to the problem

$$\tilde{\theta}_t + \left(\overrightarrow{\Phi}^* \cdot \vec{\nabla}\right)\tilde{\theta} = \lambda_0 \tilde{\theta}_{zz} + \lambda \Delta \tilde{\theta} - \frac{\eta}{\varepsilon}\tilde{\theta},$$

$$\tilde{\theta}\Big|_{t=0} = \theta^0,$$

where

$$\overrightarrow{\Phi}^* = \left(\Phi_1, \Phi_2, -\int_0^z div\overrightarrow{\Phi}dz\right).$$

The theory of boundary value problems for parabolic equations guarantees the existence and uniqueness of a solution

$$\tilde{\theta}(x, y, z, t) \in L_\infty\big(0, T; L_2(\Omega_2)\big) \cap L_2\big(0, T; W_2^1(\Omega_2)\big)$$

We use the found function $\tilde{\theta}$ to solve the system

$$\int_{\Omega_2} \{[\tilde{\Phi}_t + (\overrightarrow{\Phi}^* \cdot \nabla)\tilde{\Phi}]\vec{w}_j + \mu_0\, \tilde{\Phi}_z \vec{w}_{jz} + \mu \nabla\tilde{\Phi}\nabla\vec{w}_j + \tag{22}$$

$$+\left[\vec{\ell} \times \tilde{\tilde{\overrightarrow{\Phi}}}\right]\vec{w}_j + h\big(\tilde{\theta}\big)div\vec{w}_j + \frac{\eta}{\varepsilon}\tilde{\Phi}\vec{w}_j\Big\}\, d\Omega_2$$

Solvability of the Cauchy problem

$$\tilde{\varphi}_i(0) = \int_{\Omega_2} \tilde{\Phi}\vec{w}_i dxdydz = a_i(0),$$

This system follows the theory of ordinary differential equations. Let us denote its solution $\tilde{\varphi}(t)$.

Thus, the mapping $\Lambda: k \to C(0, T)$ is constructed. Estimate (20) guarantees that the set k is mapped into itself. Let us check that the mapping $\Lambda: k \to k$ is compact. To do this, multiply (22) by $\frac{\partial\tilde{\varphi}_j}{\partial t}$ and sum over j. We get

$$\left\|\tilde{\Phi}_t\right\|^2_{L_2(\Omega_2)} + \frac{1}{2}\frac{d}{dt}\left[\mu_0\left\|\tilde{\Phi}_z\right\|^2_{L_2(\Omega_2)} + \mu\left\|\nabla\tilde{\Phi}\right\|^2_{L_2(\Omega_2)} + \frac{\eta}{\varepsilon}\left\|\tilde{\Phi}\right\|^2_{L_2(\Omega_2)}\right] \le$$

$$\le -\int_{\Omega_2}\left\{\left[\vec{\ell} \times \tilde{\tilde{\overrightarrow{\Phi}}}\right]\tilde{\Phi}_t + (\overrightarrow{\Phi}^* \cdot \nabla)\tilde{\Phi} \cdot \tilde{\Phi}_t + \nabla h\big(\tilde{\theta}\big)\tilde{\Phi}_t\right\}dxdydz,$$

From here, we have

$$\left\|\tilde{\Phi}_t\right\|^2_{L_2(\Omega_2)} + \frac{1}{2}\frac{d}{dt}\left[\mu_0\left\|\tilde{\Phi}_z\right\|^2_{L_2(\Omega_2)} + \mu\left\|\nabla\tilde{\Phi}\right\|^2_{L_2(\Omega_2)} + \frac{\eta}{\varepsilon}\left\|\tilde{\Phi}\right\|^2_{L_2(\Omega_2)}\right] \le$$

$$\le \mathrm{C}\left(\|\Phi\|^2_{L_2(\Omega_2)} + \left\|\tilde{\Phi}^*\right\|^2_{\mathrm{C}(\Omega_2)} \times \left\|\nabla\tilde{\Phi}\right\|^2_{L_2(\Omega_2)} + \|\nabla \mathrm{h}\|^2_{L_2(\Omega_2)}\right)$$

Since the basis functions are smooth, $\left\|\tilde{\Phi}^*\right\|^2_{\mathrm{C}(\Omega_2)} \le \mathrm{C}_2$, then the inequality is true

$$\left\|\tilde{\Phi}_t\right\|^2_{L_2(\Omega_2)} \le \mathrm{C}_2$$

Thus, we have obtained that the operator Λ takes the bounded set k from $C(0,T)$ to a set from $W_2^1(0,T)$, which, by the embedding theorem, is compact in $C(0,T)$. This means that it is completely continuous, and, therefore, satisfies all the requirements of Schauder's theorem [4], and has a fixed point. From the construction it is clear that it is unique. Lemma 2 is proven.

It is easy to verify that for the approximations $\{\vec{u}_\varepsilon^N, \theta_\varepsilon^N\}$ Lemma 2 is true, i.e. there is an assessment

$$\|\vec{u}_{\varepsilon t}^N\|_{L_{\frac{4}{3}}\left(0,T;V_2^{-2}(\Omega_2)\right)} + \|\theta_{\varepsilon t}^N\|_{L_{\frac{4}{3}}\left(0,T;W_2^{-2}(\Omega_2)\right)} \le C_\varepsilon \tag{23}$$

Estimates (20), (23) guarantee that from the sequence $\{\vec{u}_\varepsilon^N, \theta_\varepsilon^N\}$ one can select sequences $\{\vec{u}_\varepsilon^N, \theta_\varepsilon^N\}$ that converge as $N \to \infty$: weakly in $\left\{L_2(0,T;\dot{V}_2^1(\Omega_2)), L_2(0,T;\dot{W}_2^1(\Omega_2))\right\}$ –,weakly in $\left\{L_\infty\left(0,T;\dot{V}_2(\Omega_2)\right),\ L_\infty\left(0,T;L_2(\Omega_2)\right)\right\}$,, strongly in $\left\{L_2\left(0,T;L_2(\Omega_2)\right), L_2\left(0,T;L_2(\Omega_2)\right)\right\}$.

These properties of the approximations allow us to go to the limit as $N \to \infty$ in identities the (8) written for $\vec{u}_\varepsilon^N, \theta_\varepsilon^N$. This means that the limit functions $\vec{u}^\varepsilon, \theta^\varepsilon$ satisfy identities (8) and, therefore, are a generalized solution to problem (5)-(7). Theorem 1 is proven.

Let us pay attention to the fact that the approximations $\{\vec{u}_\varepsilon^N, \theta_\varepsilon^N\}$, constructed in the proof of the theorem, have the same properties concerning N and ε (except for the latter). This allows us to go to the limit in integral identities as $\varepsilon \to 0$, and thus obtain that the limit $\{\vec{u}, \theta\}$ of the sequence $\vec{u}_\varepsilon^N, \theta_\varepsilon^N$ is a generalized solution to problem (2)-(4).

Let us estimate the rate of convergence of solutions as $\varepsilon \to 0$.

Let us continue the functions $\vec{u}, \theta$ into the region D_1 by zero. The functions $\vec{\varphi}, \psi$ from identities (8) satisfy the relations

Let us pay attention to the fact that the approximations $\{\vec{u}_\varepsilon^N, \theta_\varepsilon^N\}$, constructed in the proof of the theorem,

$$\int\limits_{Q_2} \left\{\vec{u}_t\vec{\varphi} + (\vec{v}\cdot\nabla)\vec{u}\vec{\varphi} + \mu_0\vec{u}_z\vec{\varphi}_z + \mu\nabla\vec{u}\nabla\vec{\varphi} + \left[\vec{\ell}\times\vec{u}\right]\varphi - \nabla h(\theta)\vec{\varphi}\right\}dxdydzdt -$$

$$-\int_0^T\int_{\partial\Omega_0}\left[\frac{\partial\vec{u}}{\partial n}\vec{\varphi} + \xi\vec{\varphi}\cdot\vec{n}\right]d(\partial\Omega_0)dt = 0 \tag{24}$$

$$\int\limits_{Q_2}\{\theta_t\psi + (\vec{v}\cdot\nabla)\theta\psi + \lambda_0\theta_z\psi_z + \lambda\nabla\theta\nabla\psi\}dxdydzdt - \int\limits_0^T\int\limits_{\partial\Omega_0}\frac{\partial\theta}{\partial n}\psi d(\partial\Omega_0)dt = 0$$

For the difference $\vec{w} = \vec{u}^\varepsilon - \vec{u}, \eta = \theta^\varepsilon - \theta$ it is true

$$\int\limits_{Q_1}\left\{\vec{w}_t\vec{\varphi} + (\vec{v}^\varepsilon\cdot\nabla)\vec{u}^\varepsilon\vec{\varphi} - (\vec{v}\cdot\nabla)\vec{u}\vec{\varphi} + \mu_0\vec{w}_z\vec{\varphi}_z + \mu\nabla\vec{w}\nabla\vec{\varphi} + \left[\vec{\ell}\times\vec{w}\right]\vec{\varphi}\right.$$

$$\left. + \nabla[h(\theta^\varepsilon) - h(\theta)]\vec{\varphi}\right\}dxdydzdt =$$

$$= \int_0^T\int_{\partial\Omega_0}\left[\frac{\partial\vec{u}}{\partial n}\vec{\varphi} + \xi\vec{\varphi}\cdot\vec{n}\right]d(\partial\Omega_0)dt, \tag{25}$$

$$\int\limits_{Q_2}\{\eta_t\psi + [(\vec{v}^\varepsilon\cdot\nabla)\theta^\varepsilon - (\vec{v}\cdot\nabla)\theta]\varphi + \lambda_0\eta_z\psi_z + \lambda\nabla\eta\nabla\psi\}dxdydzdt = \int\limits_0^T\int\limits_{\partial\Omega_0}\frac{\partial\theta}{\partial n}\psi d(\partial\Omega_0)dt = 0$$

Assuming in (25) $\vec{\varphi} = \vec{w},\ \psi = \eta$ we have

$$\frac{1}{2}\left\|\vec{W}(t)\right\|^2_{L_2(\Omega_2)}+\mu_0\|\vec{w}_z\|^2_{L_2(Q_2)}+\mu\|\nabla\vec{w}_z\|^2_{L_2(Q_2)}+\frac{1}{2}\|\eta(t)\|^2_{L_2(Q_2)}+\lambda_0\|\eta_z\|^2_{L_2(Q_2)}+\lambda\|\eta\|^2_{L_2(Q_2)}\leq \tag{26}$$

$$\leq\left|\int_{Q_2}\{(\vec{v}^\varepsilon\cdot\nabla)\vec{u}^\varepsilon\vec{w}-(\vec{v}\cdot\nabla)\vec{u}\vec{w}+\nabla[h(\theta^\varepsilon)-h(\theta)]\vec{w}\}dxdydzdt\right|+$$

$$+\left|\int_0^T\int_{\partial\Omega_0}\left[\frac{\partial\vec{u}}{\partial n}\vec{w}+\xi\vec{w}\cdot\vec{n}\right]d(\partial\Omega_0)dt\right|+\left|\int_0^T\int_{\partial\Omega_0}\frac{\partial\theta}{\partial n}\eta d(\partial\Omega_0)dt\right|+$$

$$+\left|\int_{Q_2}\{(\vec{v}^\varepsilon\cdot\nabla)\theta^\varepsilon\eta-(\vec{v}\cdot\nabla)\theta\eta\}dxdydzdt\right|$$

Let us estimate the boundary integrals

$$\left|\int_0^T\frac{\partial\vec{u}}{\partial n}\vec{w}d(\partial\Omega_0)dt\right|\leq\int_0^T\left\|\frac{\partial\vec{u}}{\partial n}\right\|_{L_2(\partial\Omega_0)}\|\vec{w}\|_{L_2(\partial\Omega_0)}dt\leq$$

$$\leq C\int_0^T\|\Delta\vec{u}\|_{L_2(\Omega_0)}\|\nabla\vec{w}\|^{\frac{1}{2}}_{L_2(\Omega_2)}\|\vec{w}\|^{\frac{1}{2}}_{L_2(\Omega_2)}dt\leq C\|\nabla\vec{w}\|^{\frac{1}{2}}_{L_2(Q_2)}\|\Delta\vec{u}\|^{\frac{1}{2}}_{L_2(Q_0)}\|\vec{w}\|^{\frac{1}{2}}_{L_2(Q_2)}\leq$$

$$\leq\frac{\delta_1}{4}\|\nabla\vec{w}\|^2_{L_2(Q_2)}+C\delta_1^{-1}\|\nabla\vec{u}\|^{\frac{4}{3}}_{L_2(Q_0)}\left[\|\vec{w}\|^{\frac{2}{3}}_{L_2(Q_0)}+\|\vec{w}\|^{\frac{2}{3}}_{L_2(Q_1)}\right]$$

$$\left|\int_0^T\int_{\partial\Omega_0}\frac{\partial\theta}{\partial n}\eta d(\partial\Omega_0)dt\right|\leq\frac{\delta_2}{4}\|\nabla\eta\|^2_{L_2(Q_2)}+C\delta_2^{-1}\|\nabla\theta\|^{\frac{4}{3}}_{L_2(Q_0)}\left[\|\eta\|^{\frac{2}{3}}_{L_2(Q_0)}+\|\eta\|^{\frac{2}{3}}_{L_2(Q_1)}\right]$$

$$\left|\int_0^T\int_{\partial\Omega_0}\xi\vec{w}nd(\partial\Omega_0)dt\right|\leq\frac{\delta_3}{4}\|\nabla\vec{w}\|^2_{L_2(Q_2)}+C\delta_3^{-1}\|\nabla\xi\|^{\frac{4}{3}}_{L_2(Q_0)}\left[\|\vec{w}\|^{\frac{2}{3}}_{L_2(Q_0)}+\|\vec{w}\|^{\frac{2}{3}}_{L_2(Q_1)}\right]$$

We estimate the integrals over the area as follows:

$$\left|\int_{Q_2}[(\vec{v}^\varepsilon\cdot\nabla)\vec{u}^\varepsilon\vec{w}-(\vec{v}\cdot\nabla)\vec{u}\vec{w}]dxdydzdt\right|=\left|\int_{Q_2}\left[(\vec{v}\cdot\nabla)\frac{\overrightarrow{w^2}}{2}+(\vec{w}\cdot\nabla)\vec{u}^\varepsilon\vec{w}\right]dxdydzdt\right|=$$

$$=\left|\int_{Q_2}(\vec{w}\cdot\nabla)\vec{u}^\varepsilon\vec{w}dxdydz\right|\leq C\|\vec{w}\|^2_{L_4(Q_2)}\|\nabla\vec{u}^\varepsilon\|^2_{L_2(Q_2)}\leq$$

$$\leq C\|\nabla\vec{w}\|_{L_2(Q_2)}\|\vec{w}\|_{L_2(Q_2)}\|\nabla\vec{u}^\varepsilon\|_{L_2(Q_2)}$$

$$\leq\frac{\delta_4}{2}\|\nabla\vec{w}\|^2_{L_4(Q_2)}+C\delta_4^{-1}\|\nabla\vec{u}^\varepsilon\|^2_{L_2(Q_2)}\left[\|\vec{w}\|^2_{L_2(Q_2)}+\|\vec{w}\|^2_{L_2(Q_1)}\right]$$

$$\left|\int_{Q_2}[(\vec{v}^{\varepsilon}\cdot\nabla)\theta^{\varepsilon}\eta-(\vec{v}\cdot\nabla)\theta\eta]dxdydzdt\right|=\left|\int_{Q_2}(\vec{w}\cdot\nabla)\theta^{\varepsilon}\eta dxdydzdt\right|\leq$$

$$\leq\frac{\delta_5}{2}\|\nabla\vec{w}\|^2_{L_2(Q_2)}+C\delta_5^{-1}\|\nabla\theta^{\varepsilon}\|^2_{L_2(Q_2)}\left[\|\vec{w}\|^2_{L_2(Q_0)}+\|\vec{w}\|^2_{L_2(Q_1)}\right]$$

$$\left|\int_{Q_2}\nabla[h(\theta^{\varepsilon})-h(\theta)]\vec{w}dxdydzdt\right|=$$

$$=\left|-\int_{Q_2}h'\left[\gamma\theta^{\varepsilon}+(1-\gamma)\theta\right]\cdot\eta\times\operatorname{div}\vec{w}dxdydzdt+\int_0^T\int_{\partial\Omega_0}h(\theta)\vec{w}\cdot nd(\partial\Omega_0)dt\right|\leq$$

$$\leq\delta_6\|\nabla\vec{w}\|^2_{L_2(Q_2)}$$

$$+C\delta_6^{-1}\left[|h'|^2\left(\|\eta\|^2_{L_2(Q_0)}+\|\eta\|^2_{L_2(Q_1)}\right)\right.$$

$$\left.+\left(\|\vec{w}\|^{\frac{2}{3}}_{L_2(Q_0)}+\|\vec{w}\|^{\frac{2}{3}}_{L_2(Q_1)}\right)|h'|^2\|\nabla\theta\|^{\frac{4}{3}}_{L_2(Q_1)}\right]$$

Thus, for sufficiently small δ_i, $i=1,2,\ldots,6$, excluding from the right side of the norm $\|\vec{w}\|_{L_2(Q_0)}$, $\|\eta\|_{L_2(Q_0)}$ we get the inequality

$$\max_{0\leq t\leq T}\left\{\|\vec{w}\|^2_{L_2(\Omega_2)}+\|\eta\|^2_{L_2(\Omega_2)}\right\}+\mu_0\|\vec{w}_z\|^2_{L_2(\Omega_2)}+\mu\|\nabla\vec{w}_z\|^2_{L_2(\Omega_2)}+\lambda_0\|\eta_z\|^2_{L_2(\Omega_2)}+$$

$$+\lambda\|\nabla\eta\|^2_{L_2(\Omega_2)}+\frac{1}{\varepsilon}\left(\|\vec{w}\|^2_{L_2(\Omega_1)}+\|\eta\|^2_{L_2(\Omega_1)}\right)$$

$$\leq C_3\left[\|\vec{w}\|^{\frac{2}{3}}_{L_2(\Omega_1)}+\|\eta\|^{\frac{2}{3}}_{L_2(\Omega_1)}+\|\vec{w}\|^2_{L_2(\Omega_1)}+\|\eta\|^2_{L_2(\Omega_1)}\right]$$

from which it follows

$$\max_{0\leq t\leq T}\left\{\|\vec{w}\|^2_{L_2(\Omega_2)}+\|\eta\|^2_{L_2(\Omega_2)}\right\}+\mu_0\|\vec{w}_z\|^2_{L_2(\Omega_2)}+\mu\|\nabla\vec{w}_z\|^2_{L_2(\Omega_2)}+\lambda_0\|\eta_z\|^2_{L_2(\Omega_2)}+$$

$$+\lambda\|\nabla\eta\|^2_{L_2(\Omega_2)}\leq C_4\left[\varepsilon^{\frac{1}{3}}+\varepsilon\right].$$

Thus, the following theorem is proven

Theorem 2. Let $\vec{u}^0(x,y,z)\in\dot{V}_2^1(\Omega_0)$, $\theta^0(x,y,z)\in W_2^1(\Omega_2)$, $\partial\Omega_0\in C^2$.. Then the following estimate holds

$$\|\vec{u}^{\varepsilon}-\vec{u}\|^2_{L_\infty(0,T;V_2(\Omega_2))}+\|\theta^{\varepsilon}-\theta\|^2_{L_\infty(0,T;L_2(\Omega_2))}+$$

$$\|\vec{u}^{\varepsilon}-\vec{u}\|^2_{L_\infty\left(0,T;\dot{V}_2^1(\Omega_2)\right)}+\|\theta^{\varepsilon}-\theta\|^2_{L_\infty\left(0,T;W_2^1(\Omega_2)\right)}\leq C_5\varepsilon^{\frac{1}{3}}$$

Remark 1. The proposed method in [6] was used for numerical calculations. The calculation results show the technique's effectiveness when the region under consideration Ω_0 has a curvilinear boundary.

Results of the study

Quantitative estimates of model accuracy

To determine the accuracy and reliability of our model, we used the following mathematical methods and statistical analyses:

1. Standard deviation according to the formula:

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - y_i)^2},$$

where x_i are the expected values of the model; y_i - observed data; n is the number of observations.

Moreover, observed values are the real ocean temperature measured at various points; predicted values are results from our ocean model for the same points.

2. Pearson correlation coefficient

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}},$$

where $\bar{x}, \bar{y}$ are the average values of the estimated and observed values, respectively.

Temperature data were taken at the end of spring in the temperate latitudes of the North Atlantic, off the European coast [1] (data are presented in Table 1).

Table 1. Temperature data

N	*Observed Temperature* (y_i, ℃)	*Predicted Temperature* (x_i, ℃)	*Error* (℃)
1	14,00	13,50	-0,50
2	15,20	14,80	-0,40
3	16,10	16,30	0,20
4	17,80	18,00	0,20
5	18,50	18,10	-0,40
6	19,00	19,40	0,40
7	20,20	20,00	-0,20
8	21,50	21,80	0,30
9	22,00	22,20	0,20
10	23,00	23,50	0,50
Standard deviation		0,349	
Pearson correlation coefficient		0,996407	

Thus, from the calculations obtained, the following conclusions can be drawn:

– the standard deviation showed that the model was in error by 0.349°C relative to the observed values, which is quite acceptable;

– the Pearson correlation coefficient between predicted and observed temperature values is 0.996, indicating a robust positive correlation. This result shows that the model reproduces observed temperatures very accurately, indicating high reliability and accuracy.

Discussion

Small-scale processes such as small-scale turbulence and internal waves significantly impact ocean dynamics and structure. These processes affect vertical and horizontal mixing, which, in turn, is critical to the accuracy of modeling parameters such as temperature, salinity, and water circulation.

Internal waves arise at the boundaries of different water densities and can transfer energy over long distances. In modeling, these waves are essential for predicting nutrient and biomass dynamics and understanding general water circulation processes. To account for internal waves in numerical models, a parameterization reproduces their effect on mixing and turbulence without the need to model each wave separately.

Our model uses the Navier-Stokes equations in the baroclinic formulation in equations (1) and (4) to describe internal waves. These equations allow us to consider the influence of changes in density caused by temperature and salt gradients on the dynamics of flows.

Small-scale turbulence plays a key role in the vertical and horizontal transport of heat, salts, and biochemicals. It is caused by viscous effects and flow instabilities and is described in our model by parameterizing the turbulent exchange of momentum and mass. The equations of motion (1) use the diffusion term $\mu\Delta u$, which models vertical and horizontal diffusion, which is important for describing turbulent processes on small scales.

The study's model integrates dynamic equations, taking into account the influence of small-scale processes through parameterization. These parameterizations allow the model to effectively reproduce the overall flow pattern without delving into each small-scale process separately. This approach helps balance computational efficiency and model accuracy.

Future versions of the model are considering introducing more complex turbulence schemes, such as turbulence kinetic energy equation (TKE) and mixed long-period scaling (LES) approaches, to improve the modeling of small-scale dynamics. These methods allow for a more accurate description of the distribution of turbulent energy and its interaction with averaged flow fields.

In addition, work remains to integrate observations of small-scale turbulent structures obtained using satellite technologies and autonomous underwater vehicles for model verification and calibration, improving the accuracy of forecasts of water mass dynamics at small scales.

Conclusion

This paper developed and analyzed a non-stationary model of baroclinic ocean motion using the fictitious domain method. The model covers key aspects of water mass dynamics, including advection processes, diffusion, Coriolis force effects, and temperature changes. The modeling results confirm the importance of considering vertical and horizontal gradients of pressure and temperature in predicting the movement of ocean currents.

The main conclusions show that the proposed model can reproduce the known characteristics of baroclinic currents with sufficient accuracy and can be used for a more detailed study of the influence of various factors on ocean dynamics. It was also demonstrated that including additional nonlinear terms in the model and considering multiple initial and boundary conditions can improve forecast quality and increase numerical schemes' stability.

However, despite the progress achieved, several limitations must be considered. In particular, further study of the influence of small-scale processes and turbulence on modeling accuracy is required. It is also important to conduct additional research into the impact of climate change on model parameters, including temperature and salt regimes.

We hope that the results of this work will serve as a basis for further developments in modeling ocean processes and contribute to an improved understanding of the complex interactions of climate and their consequences for marine and coastal ecosystems.

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