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JONSSON EXISTENTIALLY CLOSED UNARS OF EXPANDED SIGNATURE

Abstract

Being one of the important parts of fundamental mathematics, Model Theory is a young subject for modern researchers in this area. However, according to the last obtained results, this discipline will play a crucial role in the future of mathematical science. As well-known, the name "Model Theory" was introduced in 1954. It is important to distinguish that classical Model Theory introduces concepts based on considering complete theories. The given article is dedicated to the research of Jonsson theories of unars. Jonsson theories are, generally speaking, not complete. Hence, the results obtained in this article are strengthened. Firstly, we considered the theory of all unars and a class of existentially closed models of this theory. Secondly, we expanded the signature of unars that contains only one unary functional symbol by a new unary predicate and constants. Thirdly, we obtained some results concerning the universals and primitives of considered theory's existentially closed Jonsson unars. Since we are using the new methodology (so-called semantic method) for the research of Jonsson existentially closed unars. Semantic methods consist of transferring properties of fixed complete theory to considered Jonsson theory.

Keywords: model theory, Jonsson theory, semantic model, unar, existentially closed unar, universals, primitives.

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КЕҢЕЙТІЛГЕН СИГНАТУРАДАҒЫ ЙОНСОНДЫҚ ЭКЗИСТЕНЦИАЛДЫ ТҰЙЫҚ УНАРЛАР

Аңдатпа

Модельдер теориясы, фундаменталды математиканың маңызды бөліктерінің бірі бола отырып, осы саладағы заманауи зерттеушілер арасында жас бағыт болып табылады, бірақ алынған нәтижелерге сәйкес бұл бағыт математика ғылымының болашағында өте маңызды рөл атқаратын секілді. Белгілі болғандай, «модельдер теориясы» атауы 1954 жылы енгізілген. Классикалық модельдер теориясы, толық теорияларды қарастыруға негізделген ұғымдардың кіріспесі деп ажырату маңызды. Бұл мақала, унарлардың йонсондық теорияларын зерттеуге бағытталған. Йонсондық теориялар, жалпы айтқанда, толық емес болып саналады. Демек, бұл мақалада алынған нәтижелер неғұрлым нығайтылған десек те болады. Ең алдымен, барлық йонсондық унарлар теориясын және осы теорияның экзистенциалды тұйық модельдер класын қарастырдық. Екіншіден, тек жалғыз, бір орынды функционалды символдан тұратын унарлар сигнатурасын, жаңа бір орынды предикаттық және константалық символдар арқылы кеңейттік. Үшіншіден, біз қарастырылған теорияның экзистенциалды тұйық йонсондық унарларының универсалы және примитивіне қатысты кейбір нәтижелерге қол жеткіздік. Экзистенциалды тұйық йонсондық унарларды зерттеуде семантикалық әдіс деп аталатын жаңа әдіснаманы қолданып отырғандықтан, бұл әдістің негізгі идеясын ашып жазсақ - бекітілген толық теориялардың қасиеттерін қарастырылып отырған йонсондық теорияға тасымалдау болып табылады.

Түйін сөздер: модельдер теориясы, йонсондық теория, семантикалық модель, унар, экзистенциалды тұйық унар, универсалдар, примитивтер.

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ЙОНСОНОВСКИЕ ЭКЗИСТЕНЦИАЛЬНО ЗАМКНУТЫЕ УНАРЫ РАСШИРЕННОЙ СИГНАТУРЫ

Аннотация

Будучи одним из важных разделов фундаментальной математики, теория моделей является молодой темой современных исследователей в этой области, но по последним полученным результатам кажется, что эта дисциплина будет играть очень важную роль в будущем математической науки. Как известно, название "теория моделей" было введено в 1954 году. Важно отметить, что классическая теория моделей вводит понятия, основанные на рассмотрении полных теорий. Данная статья посвящена исследованию йонсоновских теорий унаров. Йонсоновские теории, вообще говоря, не являются полными. Поэтому результаты, полученные в данной статье, являются более усиленными. Во-первых, мы рассмотрели теорию всех унаров и класс экзистенциально замкнутых моделей этой теории. Во-вторых, мы расширили сигнатуру унаров, содержащих только один унарный функциональный символ, за счет нового одноместного предиката и констант. И, в-третьих, мы получили некоторые результаты, касающиеся универсалов и примитивов экзистенциально замкнутых йонсоновских унаров рассматриваемой теории. Поскольку мы используем новую методологию (так называемый семантический метод) для исследования экзистенциально замкнутых йонсоновских унаров. Семантические методы заключаются в переносе свойств фиксированной полной теории на рассматриваемую йонсоновскую теорию.

Ключевые слова: теория моделей, йонсоновская теория, семантическая модель, унар, экзистенциально замкнутый унар, универсалы, примитивы.

Main provisions

The main idea of the paper was to study properties of universals and primitives of unars in the new expanded signature. As a conclusion of the research it was proved that the expansion of one unary functional symbol signature by new constant symbol and unary predicate symbol doesn't influence the Jonssonness of the unars' theories, moreover such theories will be hereditary. As a result, the authors proved three theorems concerning: the equality of new Jonsson universal of unars and its center; the relations between new Jonsson universals, their centers and semantic models; two equivalent conditions on new Jonsson primitives of unars.

Introduction

The study of any algebraic system is strongly connected with the study of the theory that deduces the sentences true on it. The unars are one of the simple algebraic systems. In the given article, we consider a more complicated case of unars in the frame of expanded signature and three kinds of their theories: the theory of all unars, universals and primitives [1].

Since we are working in terms of Jonsson theories that are, generally speaking, not complete, we need to recall the results concerning the complete theory of unars.

Yu.Ye. Shishmaryev obtained the foundational results in this field. In 1972, Yu.Ye. Shishmaryev [2] proved three theorems concerning the complete unars theory with infinite models. The author defined the conditions that should be satisfied for the limited theory to be categorical in countable and uncountable power and for the non-limited theory to be categorical in uncountable power. In work [3], A.A. Ivanov concluded that the elementary theory of unars is decidable. In work [4], A.A. Ivanov obtained the results on strongly ultra-homogeneous unar; this result is connected with defining the criterion on the admission of quantifier elimination in the complete theory of unars, in the elementary theory of unars, as well as with the fact that every complete limited or not limited theory of unars that has an infinite model is not finitely axiomatising. A.N. Ryaskin, in work [5], counted the number of models of complete theories of unars and, in work [6], obtained the properties of the finite hull for complete unars theory. In work [7], Leo Markus obtained the criteria for the case when a model M of language L_1 is minimal and consists of prime or non-prime components. L_1 consists of a unary function symbol and a relation symbol (equality). The main theorem of [7] is based on that criteria and states that if T^1 has a minimal model which is not prime, then T^1 has $2^{\aleph^0}$ non-isomorphic minimal models. In the work [8], the author obtained beneficial results on relations between two components, i.e., the equivalence conditions and disjoint union criterion.

The study of the Jonsson theory of unars starts from the works of Professor T.G. Mustafin.

The characteristic of the semantic model of unars was obtained in the work [1] by Yeshkeyev A.R. and Mustafin T.G. In the work [9], it was proved that the Jonsson theory of unars is perfect. It is well-known that categorical Jonsson theories are perfect. With this moment, it will be interesting to pay attention to the following result regarding the categoricity phenomena of a complete theory of unars.

Corollary 1. [10] Theory of unars is ω_1-categorical if and only if it is quasi-similar to the theory of infinity sets (without any structure).

Because of the later corollary, it is necessary to note the following. In order to describe some class of particular algebraic systems defined in the corresponding model-theoretic language, there may be no characteristic in the corresponding language.

The generalisation of the theories and consideration of classes are given in the works [11, 12]. We worked in the frame of the same language of unars and with the same characteristics as in [1] and researched the behaviour of classes constructed by introducing the cosemanticness relation on the Robinson spectrum from the semantic Jonsson quasivariety of universal unars.

On the other hand, it is important to note that the theory of all unars is Jonsson theory of S-acts over cyclic monoid. In a particular case, we can consider the models of such theory in the form of the algebraic system $\{M;\ f\}_{\{f\in M\}}$, i.e. $M\times\ M\rightarrow\ M$, where M is a cyclic monoid.

$$f_e(a) = a \text{ for } e \in \mathrm{M} \text{ and all } a \in M;$$

$$f_{\{\alpha\beta\}}(a)\ =\ f_\alpha\left(f_\beta(a)\right) \text{ for all } a \in M \text{ and all } \alpha, \beta \in M.$$

As a cyclic monoid, we understand any homomorphic image of a free monoid with one generator. Obviously, any cyclic monoid is isomorphic to a cyclic group, either obtained by outer inclusion of unity to cyclic semigroup.

The conditions, when we can call a cyclic S-act free, flat or projective, are described in the work [13].

A monoid S is called a stabiliser (superstabiliser, ω-stabiliser) if $Th(A)$ is stable (superstable, ω-stable) for any S-act A over S. In the [10], it is noted that from the [8], Shelah noticed that cyclic monoids are superstabilisers.

Theorem 1. [10] A monoid S is a superstabiliser if and only if S is a quite ordered monoid.

The description of Jonsson S-acts over a group with its respective invariants of semantic models is obtained in the work [14]. There is proof that if the theory of S-acts has an infinity model, then three conditions are equivalent: the theory is inductive and has JEP (joint embedding property) and AP (amalgamation property).

In the work [15], we obtained the cosemanticness conditions of classes constructed by introducing the cosemanticness relation on the Positive Jonsson spectrum from a fixed class of S-acts over the group.

Research Methodology

Jonsson theory of unars. We will work in the frame of Jonsson theories, which are, generally speaking, not complete. Let us recall its definition.

Definition 1. [16] A theory T is said to be Jonsson, if:

1) T has at least one infinite model;
2) T is $\forall\exists$-axiomatising;
3) T has JEP property;
4) T has AP property.

Let $\mathbb{T}_U$ be the theory of all unars of given language L of the signature $\sigma =< f >$ where f is a unary functional symbol. Therefore, the theory $\mathbb{T}_U$ of all unars is empty (the axiom set of the theory is an empty set). It was noted in the work [17] that any empty theory of arbitrary signature is Jonsson theory. From this fact, we can conclude that the theory $\mathbb{T}_U$ of all unars is Jonsson theory. Nevertheless, we can prove immediately that this fact is true.

Theorem 2. The theory $\mathbb{T}_U$ of all unars is Jonsson theory.

Proof. To prove that fact, we need to use the following theorem:

Theorem 3. [17] Inductive theory T is Jonsson if and only if there is a semantic model of theory T.

Evidently, an empty theory is universal; hence, it is inductive. Let us give the definition of a semantic model of Jonsson theory.

Definition 2. [18] Let T be a Jonsson theory. A model C_T of power 2^{ω} is said to be a semantic model of the theory T if C_T is a ω^+-homogeneous ω^+-universal model of the theory T.

In other words, to prove Theorem 2, it is sufficient to construct a semantic model of the theory $\mathbb{T}_U$.

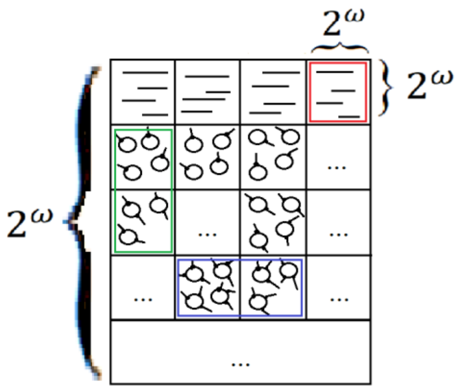

Figure 1. The semantic model of the theory $\mathbb{T}_U$ of all unars.
In colour, different models are embedded in the semantic model.

Theorem 2 is proven.

We will denote the semantic model of the theory $\mathbb{T}_U$ as $\mathfrak{C}$. Consequently, the definition of the center of Jonsson theory immediately follows from Definition 2.

Definition 3 [18] The elementary theory of a semantic model of the Jonsson theory T is called the center of this theory. The center is denoted by T^*, i.e. $Th(C) = T^*$.

Since $\mathbb{T}_U$ is Jonsson, it has its own center. Let us denote it as $\mathbb{T}_U^*$. By Definition 3, $\mathbb{T}_U^* = Th(\mathfrak{C})$ is the elementary theory of semantic model $\mathfrak{C}$ that is complete. Therefore, all the results from [2-6] apply to $\mathbb{T}_U^*$.

Let $\mathfrak{A}$ be some unar, i.e. the model of signature $\sigma =< f >$. Let $f^0(x) = x, f^{n+1}(x) = f\left(f^n(x)\right)$, $n \in \omega$. Since each model of the Jonsson theory embeds into its semantic model by Definition 2, the following fact is true.

Lemma 1. [1] For any unar $\mathfrak{A}$, the following is satisfied:

$$\mathfrak{A} \vDash T \Leftrightarrow \mathfrak{A} \text{ embeds in } \mathfrak{C}.$$

Elements $a, b \in \mathfrak{A}$ are called $\mathfrak{A}$-connected in the set $X \subseteq \mathfrak{A}$ if there exist natural numbers m and n such that $f^m(a) = f^n(b)$ and $f^0(a) = f^m(a), f^0(b), \ldots, f^n(b) \in X$.

The set X is called $\mathfrak{A}$-connected if any two elements from X are $\mathfrak{A}$-connected. A subsystem $\mathfrak{B} \subseteq \mathfrak{A}$, whose carrier is the maximal $\mathfrak{A}$-connected subset of carrier $\mathfrak{A}$, is called a component in $\mathfrak{A}$. If $\mathfrak{B}$ is a component in system $\mathfrak{A}$, then the set $\{a \in \mathfrak{B}: \mathfrak{A} \vDash (f^n(a) = a)\}$ for some $n \in \omega$ is called a cycle of a component.

Let us write down the unique connections between the elements of the component in the form of $\exists$ -formulas:

1) the property of the elements to be at "the beginning of the cycle of component": $\Phi_0^n(z) = \Phi^n(z) \& \exists y \neg \Phi(y) \& f(y) = z$, where $\Phi^n(z) = (f^n(z) = z) \& (f(z) \neq z) \dots (f^{n-1}(z) \neq z)$;

2) "x has no less than k different immediate predecessors": $\Theta_k(x) = \exists x_1, \dots, \exists x_k(\wedge_{i \neq j < k} x_i \neq x_j \wedge \wedge_{i=1}^{k} f(x_i) = x)$;

3) "there are exactly k different elements between x and the beginning of the cycle of component": $\Psi_k(x) = \exists z \exists y_1 \dots \exists y_k(\wedge_{i \neq j < x} (y_i \neq y_j) \wedge f^i(x) = y_i \wedge \wedge_{i=1}^{k-1} f(y_i) \neq f(y_{i+1}) \wedge \Phi_0^n(z) \wedge f(y_k) = z)$.

An unar is called Jonsson if it is a model of some Jonsson theory.

Lemma 2. Let $\mathbb{T}_U$ be Jonsson theory of all unars, and $\mathfrak{M}$ be its component. $\mathfrak{M}$ is a component of theory $\mathbb{T}_U$ if and only if $\mathfrak{M} \in E_{\mathbb{T}_U}$, where $E_{\mathbb{T}_U}$ is a class of existentially closed models of theory $\mathbb{T}_U$.

Proof. In order to prove the lemma, we need to use the following facts:

Theorem 5. [19] Every elementary class is the union of its components.

Let $E_{\mathbb{T}_U}$ be an elementary class of signature $\sigma =< f >$, where f is a unary functional symbol and $\mathfrak{M} \in E_{\mathbb{T}_U}$. $E_{\mathbb{T}_U}$ is a class of models of $Th(E_{\mathbb{T}_U})$, and $Th(E_{\mathbb{T}_U})$ is a set of all formulas held in $E_{\mathbb{T}_U}$.

Lemma 3. [19] Suppose $\mathcal{M}$ is an elementary class and $\mathcal{N}$ is a component of $\boldsymbol{S}\mathcal{M}$. Then, there exists $\mathfrak{A} \in \mathcal{M}$ such that $\mathcal{N} = \mathbf{0}\mathfrak{A}$.

By Lemma 3, there exists $\mathfrak{B} \in E_{\mathbb{T}_U}$ such that $\mathfrak{M}$ belongs to a universal class generated by $\mathfrak{B}$, and we will denote this class as $Mod\left(Th_{Qf}(\mathfrak{B})\right)$; $Th_{Qf}(\mathfrak{B})$ is a set of all quantifier-free formulas holding in $\mathfrak{B}$. Moreover, according to Lemma 3, we have the following fact: $Mod\left(Th_{Qf}(\mathfrak{B})\right)$ is a component of the class of substructures of $E_{\mathbb{T}_U}$. The existence of such $\mathfrak{B}$ is guaranteed by the fact that we can always consider $\mathbb{T}_U^* = Th(\mathfrak{C})$ as a set of all quantifier-free formulas holding in a semantic model $\mathfrak{C}$ of Jonsson theory of all unars.

Lemma 4. [19] Suppose $\mathcal{M}$ is an elementary class.

(i) If $\mathcal{N}$ is a component of $\boldsymbol{S}\mathcal{M}$, then $\mathcal{N} \cap \mathcal{M}$ is a component of $\mathcal{M}$ and $\boldsymbol{S}(\mathcal{N} \cap \mathcal{M}) = \mathcal{N}$.

(ii) If $\mathcal{N}$ is a component of $\mathcal{M}$, then $\boldsymbol{S}\mathcal{N}$ is a component of $\boldsymbol{S}\mathcal{M}$ and $\mathcal{N} = \mathcal{M} \cap \boldsymbol{S}\mathcal{N}$.

Therefore, by Lemma 4, $Mod\left(Th_{Qf}(\mathfrak{B})\right) \cap E_{\mathbb{T}_U}$ is a component of $E_{\mathbb{T}_U}$.

The lemma is proven. □

Jonsson universals and primitives of unars. In this section, we will work with Jonsson universals and primitives of unars. Let us recall their definitions starting from the following.

Definition 4. [1] ∇ is $\Pi_1 \cup \Sigma_1$, i.e. ∇ is a collection of all universal or existential formulas.

Definition 5. [1] 1) If $T = T_\forall$, then $T_\forall$ is said to be universal;

2) If $T = T_\nabla$, then the theory T is said to be primitive.

I.e. the theory T is universal if it consists of its universal conclusions; the theory T is primitive if it consists of its universal or existential formulas. It is easy to see that $T_\forall \subseteq T_\nabla$ and $T_\nabla = Th(\mathfrak{C})$.

The connection between two Jonsson universals concerning their centres and semantic models is presented in the following proposition.

Proposition 1. [1] Let $T_{\forall_1}$ and $T_{\forall_2}$ be Jonsson universals. Then the following conditions are equivalent:

1) $T_{\forall_1} = T_{\forall_2}$;

2) $\mathfrak{C}_{T_{\forall_1}} \simeq \mathfrak{C}_{T_{\forall_2}}$;

3) $T_{\forall_1}^* = T_{\forall_2}^*$.

As we can see, the three conditions are equivalent: two Jonsson universals are equal, their centers are equal, and their semantic models are isomorphic to each other.

Since we have already proved that the theory of unars is Jonsson, it turned out that this Jonsson theory is perfect. Let us recall the definition of perfect Jonsson theory from the work [18].

Definition 6. [18] A Jonsson theory T is called perfect if its semantic model $\mathfrak{C}$ is ω^{+}-saturated.

Consequently, we have the following theorem regarding the property of the center of perfect Jonsson theory.

Theorem 6. [20] Let T be a Jonsson theory. Then, for any model $\mathfrak{A} \in E_T$, the theory $T^0(\mathfrak{A})$ is Jonsson, where $T^0(\mathfrak{A}) = Th_{\forall\exists}(\mathfrak{A})$.

We can see that in the case of the perfectness of Jonsson theory T its center T^* is also a perfect Jonsson theory. The following theorem is a criterion of perfectness of Jonsson theory.

Theorem 7. [18] Let T be arbitrary Jonsson theory, then the following conditions are equivalent:

1) Theory T is perfect;

2) T^* is the model completion of theory T.

The following theorem is proven in the work [9] and is crucial for the main result of this section.

Theorem 8. [9] Let T be Jonsson universal of unars, and T^* be its center. Then

1) T^* is the model completion of T;

2) T^* admits quantifier elimination (i.e. submodel complete);

3) T^* is ω -stable.

Let $T_\forall$ be Jonsson universal of unars, $\mathfrak{C}_{T_\forall}$ its semantic model, and $T_\forall^*$ its center. Thus, by virtue of Theorem 7, since it was proven in work [9] that $T_\forall^*$ is a model completion of $T_\forall$, and Theorem 6 states that in this case, an arbitrary Jonsson theory is perfect, the Jonsson universal of unars $T_\forall$ is perfect Jonsson theory.

Let us consider first-order language L of the signature $\sigma =< f >$ where f is a unary functional symbol and expand it by symbols of new constant c and predicate P.

Let $\sigma'' = \sigma \cup \sigma'$, where $\sigma =< f >$, $\sigma' = (P^1, c)$. We consider a theory $\bar{T}_\forall$ in the new expanded signature σ'' as follows:

$$\bar{T}_\forall = T_\forall \cup Th_\forall\left(\mathfrak{C}_{T_\forall}, a\right)_{a \in P^1\left(\mathfrak{C}_{T_\forall}\right) \cup P^1(c)} \cup \{P^1, \subseteq\} \cup P^1(c).$$

Here, P^1 is a new unary predicate symbol, $\{P^1, \subseteq\}$ is an infinite set of sentences, which express the fact that in $\mathfrak{C}_{T_\forall}$ the predicate P^1 distinguishes existentially closed submodel of $\mathfrak{C}_{T_\forall}$, i.e. $P^1\left(\mathfrak{C}_{T_\forall}\right) = \mathfrak{M}$, $\mathfrak{M} \in E_{T_\forall}$, $\mathfrak{M}$ is an existentially closed model (Jonsson existentially closed unar), $E_{T_\forall}$ is a class of existentially closed models of theory $T_\forall$.

The existence of such structure $\mathfrak{M}$ is shown according to the Tarski-Vaught Test. The test states that such elementary extension $\mathfrak{B}$ exists for substructure $\mathfrak{A}$ that $\mathfrak{A} \preccurlyeq \mathfrak{B}$. Hence, $\mathfrak{A} \preccurlyeq_{\exists_1} \mathfrak{B} \Leftrightarrow \mathfrak{A}$ is existentially closed in $\mathfrak{B}$. Hence, $\mathfrak{M} \preccurlyeq_{\exists_1} \mathfrak{C}_{T_\forall}$.

Let us consider whether the new theory $\bar{T}_\forall$ in the newly expanded signature will be a Jonsson theory. The following definition may be useful.

Definition 7. [21] A Jonsson theory is said to be hereditary if, in any of its permissible expansion, it preserves the Jonssonness.

Let us consider $\bar{T}_\forall$ as it was described above.

Theorem 9. If Jonsson theory of unars $T_\forall$ is perfect Jonsson theory, $\bar{T}_\forall$ is its hereditary expansion, then $\bar{T}_\forall$ is also perfect Jonsson theory of unars.

Proof. Let $T_\forall$ be perfect Jonsson theory of unars, and $\mathfrak{C}_{T_\forall}$ is its semantic model. We introduce the permissible expansion into the original signature as described above and obtain a new theory denoted as $\bar{T}_\forall$. By the work [22], the expansion is permissible when it is concluded by a predicate that distinguishes an existentially closed model. Therefore, by Definition 11, the $\bar{T}_\forall$ is, in fact, Jonsson theory. Hence, a semantic model of theory $\bar{T}_\forall$ exists according to Theorem 3, which we will denote as $\bar{\mathfrak{C}}_{T_\forall}$.

Let us denote the center of $\bar{T}_\forall$ as follows:

$$\bar{T}^*_\forall = Th\left(\bar{\mathfrak{C}}_{T_\forall}\right) = Th\left(\mathfrak{C}_{T_\forall}, c, a\right)_{c,a\in P^1\left(\mathfrak{C}_{T_\forall}\right)}$$

By Definition 6, we have that the semantic model $\mathfrak{C}_{T_\forall}$ of theory $T_\forall$ is ω^+ -saturated.

Hence, $\mathfrak{C}_{T_\forall} \vDash p_c$, where p_c is a main type consisting of formulas with new constants c.

Suppose $\bar{T}_\forall$ is not perfect Jonsson theory; hence, $\bar{\mathfrak{C}}_{T_\forall}$ is not ω^+ -saturated. It means there exists a type $b \vDash p_X$, $X \in \bar{\mathfrak{C}}_{T_\forall}$, $|X| \leq \omega^+$, $b \in X$ such that $\bar{\mathfrak{C}}_{T_\forall} \nvDash p_X$. Thus, it will be realised in some elementary expansion $\bar{\mathfrak{C}}'_{T_\forall}$. If we restrict $\bar{\mathfrak{C}}'_{T_\forall}/\sigma'$, we get $\mathfrak{C}'_{T_\forall}$, which is an elementary expansion of $\mathfrak{C}_{T_\forall}$ such that $\mathfrak{C}'_{T_\forall} \vDash p_X$. I.e. $b \in \mathfrak{C}'_{T_\forall}$, however, $b \notin \mathfrak{C}_{T_\forall}$. $\mathfrak{C}_{T_\forall}$ is $the\ \omega^+$-saturated model since $T_\forall$ is the perfect Jonsson theory of unars. Therefore, $\mathfrak{C}_{T_\forall} \vDash p_X$ and $b \in \mathfrak{C}_{T_\forall}$. The same elements will realise the same type. We get the contradiction.

Hence, $\bar{\mathfrak{C}}_{T_\forall}$ is also ω^+ -saturated, and $\bar{T}_\forall$ is perfect Jonsson theory of unars. The theorem is proven.

Let $\sigma'' = \sigma \cup \sigma'$, where $\sigma =< f >, \sigma' = (P^1, c)$. We consider a theory $\bar{T}_\nabla$ in the new expanded signature σ'' as follows:

$$\bar{T}_\nabla = T_\nabla \cup Th_\nabla\left(\mathfrak{C}_{T_\nabla}, a\right)_{a\in P^1\left(\mathfrak{C}_{T_\nabla}\right)\cup P^1(c)} \cup \{P^1, \subseteq\} \cup P^1(c).$$

Here, P^1 is a new unary predicate symbol, $\{P^1, \subseteq\}$ is an infinite set of sentences, which express the fact that in $\mathfrak{C}_{T_\nabla}$ the predicate P^1 distinguishes existentially closed submodel of $\mathfrak{C}_{T_\nabla}$, i.e. $P^1\left(\mathfrak{C}_{T_\nabla}\right) = \mathfrak{M}$, $\mathfrak{M} \in E_{T_\nabla}$, $\mathfrak{M}$ is an existentially closed model (Jonsson existentially closed unar), E_{T_∇} is a class of existentially closed models of theory T_∇.

It is easy to see that $\bar{T}_\nabla \supseteq \bar{T}_\forall$ and Theorem 9 applies to considered $\bar{T}_\nabla$. Hence, $\bar{T}_\nabla$ is the perfect Jonsson primitive of unars; we will denote its center as $\bar{T}^*_\nabla$ and its semantic model as $\bar{\mathfrak{C}}_{T_\nabla}$.

Results of the study

The following theorems were obtained.

Theorem 10. Let $\bar{T}_\nabla$ be Jonsson primitive of unars, $\bar{T}^*_\nabla$ be its center, then $\bar{T}_\nabla = \bar{T}^*_\nabla$.

Proof. Since $\bar{T}_\nabla \subseteq \bar{T}^*_\nabla$, then it is obvious that $\bar{T}_\nabla \subseteq \bar{T}^*_\nabla$. Let us prove the inverse inclusion by the contradiction. Suppose that $\varphi \in \bar{T}^*_\nabla \backslash \bar{T}_\nabla$. Let $\varphi = \forall \bar{x} \psi(\bar{x})$. Since $\bar{T}_\nabla \vdash \varphi$ is incorrect, then $\bar{T}_\nabla \cup \{\neg\varphi\}$ is a consistent theory. Let $\mathfrak{A} \vDash \bar{T}_\nabla \cup \{\neg\varphi\}$. Then $\mathfrak{A} \vDash \exists \bar{x} \neg\psi(\bar{x}), \mathfrak{A} \vDash \bar{T}_\nabla$. Due to the ω^+ $-$universality of the model $\bar{\mathfrak{C}}_{T_\nabla}$, we can assume that $\mathfrak{A} \subseteq \bar{\mathfrak{C}}_{T_\nabla}$, where $\bar{\mathfrak{C}}_{T_\nabla}$ is the semantic model of $\bar{T}_\nabla$ and $\bar{\mathfrak{C}}_{T_\nabla} \vDash \bar{T}^*_\nabla$. Let $\bar{a} \in \mathfrak{A}$ such that $\mathfrak{A} \vDash \neg\psi(\bar{a})$. Since the formula $\neg\psi(\bar{x})$ contains no quantifiers, $\bar{\mathfrak{C}}_{T_\nabla} \vDash \neg\psi(\bar{a})$. However, $\varphi \in \bar{T}^*_\nabla$ and $\bar{\mathfrak{C}}_{T_\nabla} \vDash \bar{T}^*_\nabla$, so $\bar{\mathfrak{C}}_{T_\nabla} \vDash \varphi$, that is, $\bar{\mathfrak{C}}_{T_\nabla} \vDash \forall \bar{x} \psi(\bar{x})$. We have a contradiction.

The theorem is proved. □

Theorem 11. Let $\bar{T}_{\forall_1}$, $\bar{T}_{\forall_2}$ be Jonsson universals of unars, $\bar{\mathfrak{C}}_{T_{\forall_1}}, \bar{\mathfrak{C}}_{T_{\forall_2}}$ be their semantic models, and $\bar{T}_{\forall_1}^*$, $\bar{T}_{\forall_2}^*$ be their centers correspondingly. Then the following conditions are equivalent:

1) $\bar{T}_{\forall_1} = \bar{T}_{\forall_2}$;

2) $\bar{\mathfrak{C}}_{T_{\forall_1}} \simeq \bar{\mathfrak{C}}_{T_{\forall_2}}$;

3) $\bar{T}_{\forall_1}^* = \bar{T}_{\forall_2}^*$.

Proof. 1) $\Rightarrow$ 2) $\Rightarrow$ 3) trivial.

3) $\Rightarrow$ 1). Using Theorem 10, we have $\bar{T}_{\forall_1} = \bar{T}_{\forall_1}^* = \bar{T}_{\forall_2}^* = \bar{T}_{\forall_2}$. □

Lemma 5. Let $\bar{T}_\nabla$ be a Jonsson theory, $\bar{T}^*_\nabla$ be its center, and $\bar{T}'_\nabla$ be such a theory that $\bar{T}_\nabla \subseteq \bar{T}'_\nabla \subseteq \bar{T}^*_{\forall\exists}$. Then $\bar{T}'_\nabla$ is also a Jonsson theory.

Proof. It is easy to see that $\mathfrak{C}_{T_\nabla}$ is also a $\overline{T}'_\nabla$-universal $\overline{T}'_\nabla$-homogeneous model of the $\forall\exists$-theory of $\overline{T}'_\nabla$. Hence, $\overline{T}'_\nabla$ is Jonsson.

The lemma is proved. □

Theorem 12. Let $\overline{T}_\nabla$ be a Jonsson primitive of unars. Then the following conditions are equivalent:

1) $\overline{T}_\nabla$ is a maximal Jonsson primitive of unars;

2) The theory of $\overline{T}_\nabla$ is complete with respect to ∇.

Proof. 1) $\Rightarrow$ 2). Let $\varphi \in \nabla$. Suppose that $\overline{T}_\nabla \vdash \varphi$ is false and $\overline{T}_\nabla \vdash \neg\varphi$ is false. Furthermore, let ψ be one of the formulas $\varphi, \neg\varphi$ such that $\overline{T}_\nabla \subsetneq \overline{T}_\nabla \cup \{\psi\} \subseteq \overline{T}^*_{\forall\exists}$. By Lemma 6, $\overline{T}_\nabla \cup \{\psi\}$ is a Jonsson primitive theory. It contradicts the maximality of $\overline{T}_\nabla$.

2) $\Rightarrow$ 1). Let $\varphi \in \nabla$ and $\overline{T}_\nabla \cup \{\varphi\}$ be consistent by the completeness of $\overline{T}_\nabla$ with respect to ∇ we have $\overline{T}_\nabla \vdash \varphi$. Then, $\overline{T}_\nabla$ is a maximal primitive.

The theorem is proved. □

Proposition 2. 1) Every Jonsson universal of unars $\overline{T}_\forall$ is complete with respect to ∇ and is a maximal universal.

2) There exists a maximal Jonsson universal $\overline{T}_\forall$ which is not ∇-complete.

Proof. 1) The proof is the same as 2) $\Rightarrow$ 1) of Theorem 12.

2) Let σ be an empty signature, and $\overline{T}_\forall$ be the theory of all models of this signature. Obviously, $\overline{T}_\forall$ is the only, and hence maximal, universal Jonsson theory. However, $\overline{T}_\forall$ is not ∇ complete since $\overline{T}_\forall \vdash \exists xy(x \neq y)$ is false, and $\overline{T}_\forall \vdash \forall xy(x = y)$ is also false.

The proposition is proved. □

Discussion

The article does not consider positive Model Theory in terms of studying Jonsson theories. However, the authors are interested in researching considered unars and corresponding S-acts in the frame of positive Jonsson theories. One can consider the works [23, 24] to research a given field. As well as it is of interest to research the hereditary Jonsson theories of unars in the newly expanded signature in terms of consideration of their Jonsson spectrum and semantic Jonsson quasivariety.

Conclusion

Despite a Jonsson unar being the simplest algebraic system, obtained results play an important role in the research of Jonsson theory. The article proves several significant facts, such as:

1) The theory of all unars is Jonsson theory;

2) The theory of all unars and the new universal of unars in the expanded signature coincide with their respective centers;

3) A component of the semantic model of the theory of all unars is an existentially closed Jonsson unar;

4) The newly obtained theories of the newly expanded signature granted that expansion is permissible are hereditary Jonsson theories.

Besides the listed results, we have proved in which cases new primitives and new universals are complete with respect to the set of all existential or all universal sentences and found the properties of equality of two universals with respect to their semantic models and centers.

Obtained results serve as a foundation for researching the unars in terms of positive Jonsson theories, their Jonsson spectrum and semantic Jonsson quasivariety, and considering an unar as an S-act over cyclic monoid.

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