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INEQUALITY OF THE BEST ANGULAR APPROXIMATION

Abstract

The aim of the study is to analyze the best angular approximations of functions of many variables and to estimate these approximations from above. This allows one to better understand the applicability limits of these methods and optimize their use in problems of mathematical modeling and data analysis. In this paper, the properties of functions of several variables were studied and approaches to their approximation using angular approximations were proposed. Generalized Liouville–Weyl derivatives, which are used instead of classical mixed Weyl derivatives, are considered. The concept of general monotone sequences plays an important role in the study. The methodology includes the use of functional analysis, approximation theory, and analysis of parameters affecting the accuracy of approximation. The main results of the work are to obtain upper bounds for angular approximations for generalized Liouville–Weyl derivatives depending on the characteristics of the approximated functions. The results obtained are of interest to theorists working in the field of multivariate data analysis and mathematical modeling. The topic discussed in the article is closely related to the articles by A.A. Konyushkov, S.B. Stechkin, M.F. Timan, M.K. Potapova, B. Simonov, S. Tikhonov.

Keywords: Lebesgue space, best approximation with multidimensional angle, trigonometric polynomial, generalized Liouville-Weil derivative, general monotone sequences.

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ЕҢ ЖАҚЫН БҰРЫШТЫҚ ЖУЫҚТАУДЫҢ ТЕҢСІЗДІГІ

Аңдатпа

Зерттеудің мақсаты – көпт айнымалы функциялардың ең жақсы бұрыштық жуықтауларын талдау және осы жуықтаулардың жоғарыдан бағалау. Бұл осы әдістердің қолданылу шегін жақсырақ түсінуге және оларды математикалық модельдеу және деректерді талдау мәселелерінде пайдалануды оңтайландыруға мүмкіндік береді. Жұмыстың бір бөлігі ретінде көп айнымалы функцияларының қасиеттері зерттелді және бұрыштық жуықтауларды қолдану арқылы оларды жуықтау тәсілдері ұсынылды. Классикалық дербес Weyl туындыларының орнына қолданылатын жалпыланған Лиувиль-Вейль туындылары қарастырылады. Зерттеуде жалпы монотонды тізбектер туралы түсінік маңызды рөл атқарады. Әдістеме функционалдық талдауды қолдануды, жуықтау теориясын және жуықтау дәлдігіне әсер ететін параметрлерді талдауды қамтиды. Жұмыстың негізгі нәтижелері жуықталатын функциялардың сипаттамаларына байланысты жалпыланған Лиувиль-Вейль туындылары үшін бұрыштық жуықтаулардың жоғарғы шекараларын алудан тұрады. Алынған нәтижелер көпөлшемді деректерді талдау және математикалық модельдеу саласында жұмыс істейтін теоретиктерді қызықтырады. Мақалада қарастырылған тақырып А.А. Конюшков, С.Б. Стечкин, М.Ф Тиман, М.К. Потапов, Б.Симонов, С.Тихонов мақалаларымен тығыз байланысты.

Түйін сөздер: Лебег кеңістігі, көп өлшемді бұрышпен ең жақын жуықтау, тригонометриялық көпмүшелік, жалпыланған Лиувиль-Вейль туындысы, жалпы монотонды тізбек.

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НЕРАВЕНСТВО НАИЛУЧШЕГО УГЛОВОГО ПРИБЛИЖЕНИЯ

Аннотация

Целью исследования является анализ наилучших угловых приближений функций многих переменных и оценить эти приближения сверху. Это позволяет лучше понять границы применимости данных методов и оптимизировать их использование в задачах математического моделирования и анализа данных. В рамках работы были изучены свойства функций многих переменных и предложены подходы к их аппроксимации с использованием угловых приближений. Рассматриваются обобщенные производные Лиувилля–Вейля, которые используются вместо классических смешанных производных Вейля. Важную роль в исследовании играет понятие общих монотонных последовательностей. Методология включает применение функционального анализа, теории приближений и анализа параметров, влияющих на точность аппроксимации. Основные результаты работы заключаются в получении верхних оценок для угловых приближений для обобщенных производных Лиувилля–Вейля в зависимости от характеристик аппроксимируемых функций. Полученные результаты представляют интерес для теоретиков, работающих в области анализа многомерных данных и математического моделирования. Тема, обсуждаемая в статье, тесно связана со статьями А.А. Конюшкова, С.Б. Стечкина, М.Ф. Тимана, М.К. Потаповой, Б. Симонова, С. Тихонова.

Ключевые слова: пространство Лебега, наилучшее приближение с многомерным углом, тригонометрический полином, обобщенная производная Лиувилля-Вейля, общие монотонные последовательности.

Basic provisions

Theories of approximation by "angle" of functions of many variables. This approximation method gave an answer to the topical question about the constructive characteristics of classes of functions that have a given mixed smoothness. The method was greatly developed and applied in the works of other mathematicians. Estimating the best approximation by an angle in a multidimensional case is related to the theory of approximation of functions and the geometry of vector spaces. The problem of best approximation by angle is related to principal component analysis, machine learning methods and optimization problems in convex sets.

Introduction

Modern approximation theory of functions of many variables plays an important role in mathematics and its applications, including data processing, modeling of physical processes and machine learning. The study of optimal approximations of functions is an important task, since it allows not only to reduce computing resources, but also to increase the accuracy of solving problems in various fields of science and technology.

In the context of a growing number of multidimensional problems requiring computational approaches, the study of angular approximations of functions is especially relevant. Such approximations provide an effective representation of complex objects with minimal deviations, which is important for analysis, forecasting and optimization. However, many issues related to the accuracy and limits of angular approximations remain insufficiently studied.

In this paper, an analysis of the best angular approximations of functions of many variables is carried out using modern methods of functional analysis. Upper estimates of the best angular approximations for the generalized Liouville-Weyl derivatives are obtained. The main goal of the study is to develop a theoretical basis for estimating angular approximations of functions of many variables and to identify factors affecting the effectiveness of these methods. The following hypotheses were tested in the study: The best angular approximations for a class of functions of many variables can be presented with upper bounds that depend on the parameters of the function. This study lays a theoretical basis for further study of angular approximations and also offers practical recommendations for their application in real problems.

Let $f \in L_p([0,2\pi]^n)$ be the set of measurable functions of n variables $f(x_1,\dots,x_n)$, 2π- periodic in each variable, for which

$$||f||_p = \left(\int_0^{2\pi} \dots \int_0^{2\pi} |f(x_1,\dots,x_n)|^p dx_1 \dots dx_n\right)^{\frac{1}{p}} < \infty, \qquad 1 \le p < \infty.$$

Let L_p^0 be the space of functions $f \in L_p([0,2\pi]^n)$ such that $\int_0^{2\pi} f(x_1,\dots,x_n)dx_i = 0$ for almost al $x_1,\dots,x_{i-1},x_{i+1},\dots,x_n$ and for all $i = 1,2,\dots,n$.

Let $Y_{l_{i_1}\dots l_{i_m}}(f)_p$ be the best approximation of the m-dimensional angle of the function f in variables $x_{i_1},\dots,x_{i_m}$ [1],

$$Y_{l_{i_1}\dots l_{i_m}}(f)_p = \inf_{T_{l_{i_j}}} \|f - \sum_{j=1}^{m} T_{l_{i_j}}\|_p, \;\; l_{i_j} = 0,1,2\dots,$$

where $T_{l_{i_j}}$ is a trigonometric polynomial of order at most l_{i_j} ($l_{i_j} \in \mathbb{N} \cup \{0\}$) with respect to x_{i_j} and such that $T_{l_{i_j}} \in L_p([0,2\pi]^n)$, $1 \le i_j \le n$, $1 \le j \le m \le n$.

The work is devoted to obtaining new upper bounds on the norm and best angular approximations of functions of several variables with a generalized Liouville−Weil derivative through the best angular approximations of the original functions.

In the one-dimensional case, inequalities for norms and best approximations were studied by many authors, such as Konyushkov [2], Stechkin [3], Timan [4]. In [5], [6], and [7], the authors obtained inequalities for the norms and best approximations for the generalized Liouville-Weil derivatives. In [8], the authors proved the inequality for the best approximation by a twodimensional angle. Our main goal is to obtain an inequality for the best approximation by an n dimensional angle.

Let $\sigma(f)$ be the Fourier series of the functions $f \in L_p([0,2\pi])^n$ i.e.

$$\sigma(f) := \sum_{\nu_1=-\infty}^{\infty} \dots \sum_{\nu_n=-\infty}^{\infty} c_{\nu_1\dots\nu_n} e^{i\sum_{j=1}^{n} \nu_j x_j}, \tag{1}$$

where

$$c_{\nu_1\dots\nu_n} = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} f(t_1\dots t_n) e^{-i\sum_{j=1}^{n} t_j \nu_j} dt_1 \dots dt_n.$$

The transformed Fourier series of $\sigma(f)$ is given by

$$\sigma(f,\lambda,\beta_1\dots\beta_n) \equiv \sum_{\nu_1=-\infty}^{\infty} \dots \sum_{\nu_n=-\infty}^{\infty} \lambda_{\nu_1\dots\nu_n} c_{\nu_1\dots\nu_n} e^{i(\sum_{j=1}^{n} \nu_j x_j + \beta_j \frac{\pi}{2})} \mid \nu_1 \mid^{\beta_1} \dots \mid \nu_n \mid^{\beta_j}, \tag{2}$$

where $\beta_1,\dots,\beta_j \in R^n$, $\lambda = \{\lambda_{\nu_1,\dots\nu_n}\}_{\nu_1,\dots\nu_n \in N}$ is a sequence of positive numbers.

Let $f^{(\alpha)}(x_1\dots x_n)$ be the Weyl derivative of the function $f(x)$ of order $\alpha = (\alpha_1,\dots,\alpha_n), (\alpha_1 \ge 0,\dots,\alpha_n \ge 0)$.

Let $f(x) \in L_p$ and

$$f^{\alpha}(x) \sim \sum c_k (ik)^{\alpha} e^{ikx},$$

where $(ik)^{\alpha} = (ik_1)^{\alpha_1}\dots(ik_n)^{\alpha_n}$. If $\alpha_j = 0$ for some j, then this means that x_j differentiation is not performed and the corresponding factor in this product is equal to $(ik_j)^0 = 1$. When $k_j \neq 0$ this automatically takes place. If $k_j = 0$, then in this case it is assumed to be equal to $0^0 = 1$. With non-integer numbers α the expression $(it)^{\alpha}$, where t is a real value, is understood as follows: $(it)^{\alpha} = |t|^{\alpha} exp\{\frac{i\pi\alpha}{2} sign\, t\}$. Thus the transformed Fourier series (2) is a generalization of the Weyl fractional derivative.

Let's determine the difference

$$\Delta_i\lambda_{k_1,\dots k_i,\dots,k_n} = \lambda_{k_1,\dots k_{i+1},\dots,k_n} - \lambda_{k_1,\dots k_i,\dots,k_n}.$$

Let us define a generalized monotonic sequence in the n-dimensional case.

Definition 1. [9,10] A sequence $\lambda := \{\lambda_n\}_{n=1}^{\infty}$ is said to be general monotone, written $\lambda \in \mathrm{GM}^n$, if the relations

$$\sum_{m_1=k_1}^{2k_1}\left|\Delta_1\lambda_{m_1,k_1,k_2,\dots,k_n}\right| \le C\left|\lambda_{m_1,k_1,k_2,\dots,k_n}\right|,\dots,\sum_{m_1=k_n}^{2k_n}\left|\Delta_1\lambda_{m_1,k_1,k_2,\dots,m_n}\right| \le C\left|\lambda_{m_1,k_1,k_2,\dots,k_n}\right|,$$

$$\sum_{m_{n-1}=k_1}^{2k_{n-1}}\sum_{m_1=k_n}^{2k_n}\left|\Delta_{n-1}(\Delta_n\lambda_{k_1,\dots,m_{n-1},m_n}\right| \le C\left|\lambda_{k_1,k_2,\dots,k_n}\right|, \dots$$

$$\sum_{m_1=k_1}^{2k_1}\dots\sum_{m_n=k_n}^{2k_n}\left|\Delta_1(\Delta_2\dots(\Delta_n\lambda_{m_1,\dots,m_n}))\right| \le C \mid \lambda_{k_1,k_2,\dots,k_n} \mid$$

hold for all integer $k_1,k_2,\dots,k_n$ where the constant C is independent of $k_1, k_2,\dots,k_n$.

For non-negative functionals $F(f,\delta)$ and $G(f,\delta)$ we write that $F(f,\delta) \lesssim G(f,\delta)$, if there exists a positive constant C, independent of f and δ such that $F(f,\delta) \le CG(f,\delta)$. If \$ $F(f,\delta) \lesssim G(f,\delta)$ and $G(f,\delta) \lesssim F(f,\delta)$, then we will write that $F(f,\delta) \asymp G(f,\delta)$.

Research methodology

The research was conducted at the Faculty of Mechanics and Mathematics of the Eurasian National University, specializing in mathematical research and approximation theory. The results of previous studies were used in the course of the work, including the works of Potopov, as well as modern approaches to estimating the best approximations of multidimensional functions. Particular attention was paid to generalized derivatives in the sense of Liouville-Weyl and generalized monotone sequences. The research methods included the use of various inequalities from the theory of functional analysis, which are written as a lemma from below. These methods made it possible to establish upper bounds for the angular best approximations of functions of several variables.

In this section, we give some notations and lemmas which will be used in the proof of our main results. Let $\overline{l} = (l_1,\dots,l_n)$ be an element of n-dimensional space with positive coordinates and a non-empty set $e \in e_n$. Let

$$G_{\overline{l}}(e) = \{\overline{k} = (k_1,\dots,k_n) \in Z^n\colon |k_j| \le l_j, j \in e, |k_j| > l_j, j \notin e\}.$$

Let us consider the partial sums of the Fourier series of the function $f \in L_p(\mathbb{T}^d)$ in various variables $S_l(f,\overline{x}) = S_{l_1,\dots,l_n}(f,\overline{x}) = \sum_{|\nu_1|\le l_1}\dots\sum_{|\nu_n|\le l_n} c_{\overline{\nu}}(f)e^{i<\overline{\nu},\overline{x}>}$ is the partial sum over all variables, $S_{l_1,\infty}(f,\overline{x}) = \sum_{|\nu_1|\le l_1}\sum_{\nu_2=-\infty}^{+\infty}\dots\sum_{\nu_n=-\infty}^{+\infty} c_{\overline{\nu}}(f)e^{i<\overline{\nu},\overline{x}>}$ is the partial sum over the variable x_1. In a more general case

$$S_{l^e,\infty}(f,\overline{x}) = \sum_{\nu\in\Pi_{j\in e}[-l_j,l_j]\times R^{n-|e|}} c_{\overline{\nu}}(f)e^{i<\overline{\nu},\overline{x}>}$$

is the partial sum over variables x_j for $j \in e$. For a given subset $e \in e_n$ we set

$$U_{\bar{l}}(f,\bar{x}) = \sum_{e\in e_n, e\neq\emptyset} \sum_{\bar{k}\in G_{\bar{l}}(e)} c_{\bar{v}}(f) e^{i<\bar{v},\bar{x}>} i.$$

In particular, for $n = 2$ [1] we have $U_{(l_1,l_2)}(f,\bar{x}) = S_{l_1,\infty}(f,\bar{x}) + S_{\infty,l_2}(f,\bar{x}) - S_{l_1,l_2}(f,\bar{x})$. For $\mu \in Z^n$, we set $\rho(\mu) = \{\nu \in Z^n: [2^{\mu_j - 1}] \leq |\nu_j| < 2^{\mu_j}\}$, where [a] denotes an integer a and

$$D_\mu(f) := \sum_{\nu\in\rho(\mu)} c_{\bar{v}}(f) e^{i<\bar{v},\bar{x}>}.$$

Lemma 1. *[11-13]* A sequences $\{\lambda_m\} \in GM$ if and only if there exists $C > 0$, such that

$(i) \mid \lambda_m \mid \leq C \mid \lambda_k \mid$ for $k \leq m \leq 2k$,
$(ii) \sum_{m=k}^{N} \mid \Delta\lambda_m \mid \leq C(\mid \lambda_k \mid + \sum_{m=k+1}^{N} \frac{|\lambda_m|}{m})$ for any $k < N$.

From [11,12] it follows that if $\{\lambda_{m_1,\dots,m_n}\} \in GM^n$, then $\mid \lambda_{m_1,\dots,m_n} \mid \leq C \mid \lambda_{k_1,\dots,k_n} \mid$ for $k_i \leq m_i \leq 2k_i, i = 1,\dots n$.

Lemma 2. [14] *Let* $a_n \geq 0, 0 < \alpha \leq \beta < \infty$. Then

$$(\sum_{\nu=1}^{\infty} a_\nu^\beta)^{\frac{1}{\beta}} \leq (\sum_{\nu=1}^{\infty} a_\nu^\alpha)^{\frac{1}{\alpha}}).$$

Lemma 3. (Minkowski inequality,[14]) Let $1 \leq p < \infty$ and $a_{\nu k} \geq 0$. Then

(a) $(\sum_{k=1}^{\infty} (\sum_{\nu=1}^{k} a_{\nu k})^p)^{\frac{1}{p}} \leq \sum_{\nu=1}^{\infty} (\sum_{k=\nu}^{\infty} a_{\nu k}^p)^{\frac{1}{p}}$,

(b) $(\sum_{k=1}^{\infty} (\sum_{\nu=k}^{\infty} a_{\nu k})^p)^{\frac{1}{p}} \leq \sum_{\nu=1}^{\infty} (\sum_{k=1}^{\nu} a_{\nu k}^p)^{\frac{1}{p}}$.

Lemma 4. [14] For a function $f(u,y)$ defined on a measurable set $E = E_1 \times E_2 \subset \mathbb{R}_n$, where $x = (u,y), u = (x_1,\dots,x_m), y = (x_{m+1},\dots,x_n)$, the following inequality holds

$$(\int_{E_1} \mid \int_{E_2} f(u,y)dy \mid^p du)^{\frac{1}{p}} \leq \int_{E_2} (\int_{E_1} \mid f(u,y) \mid^p du)^{\frac{1}{p}} dy.$$

Lemma 5. [15] Let $f \in L_p(\mathbb{T}^n)$, $1 < p < \infty$, $m_i \in \mathbb{N} \cup 0 (i = 1,2)$. Then

$$Y_{2^{k_1}-1,\dots,2^{k_m}-1}(f)_p \leq \|f - U_{2^{k_1}-1,\dots,2^{k_m}-1}\|_p \leq \sum_{\nu_m=k_m}^{\infty} \dots \sum_{\nu_1=k_1}^{\infty} \| \sum_{s_m=2^{\nu_m}+1}^{2^{\nu_m+1}} \dots \sum_{s_1=2^{\nu_1}+1}^{2^{\nu_1+1}} D_s(f)\|_p.$$

Lemma 6. [14]
Let $1 < p < \infty$ and (1) the Fourier series for $f \in L_{p^0}(\mathbb{T}^n)$, then

$$C_1\|f\|_p \leq \| \sum_{s\in Z_+^m} |D_s(x)|^2 \|_p^{\frac{1}{2}} \leq C_2\|f\|_p,$$

generalizing the Littlewood-Paley inequalities to the multidimensional case, where $C_1, C_2 > 0$ are independent of f.

Results of the study

Let us formulate the main results.

Theorem 1. *Let* $1 < p < \infty$, $0 < \theta \leq \min(p, 2)$, $\lambda := \{\lambda_{k_1,k_2,\dots,k_n}\}_{k_1,k_2,\dots,k_n \in N}$ be sequence of positive numbers satisfying $\lambda \in GM^n$, $\alpha_i \in \mathbb{R}_+$, $r_i \in \mathbb{R}_+ \cup \{0\}$ and $\beta_i \in \mathbb{R}(i = 1,2)$. If for $f \in L_p^0(\mathbb{T}^n)$ and

$$\sum_{k_1=1}^{\infty} (\Delta_1 \lambda_{k_1,1,1\dots,1}^{\theta}) Y_{k_1,0,\dots,0}^{\theta}(f)_p + \dots + \sum_{k_n=1}^{\infty} (\Delta_n \lambda_{1,1,\dots,k_n}^{\theta}) Y_{0,0,\dots,k_n}^{\theta}(f)_p$$
$$+ \sum_{k_1=1}^{\infty} \sum_{k_n=1}^{\infty} (\Delta_{n-1}(\Delta_n \lambda_{1,\dots,k_{n-1},k_n}^{\theta})) Y_{0,0,\dots,k_{n-1},k_n}^{\theta}(f)_p \qquad (3)$$
$$+ \dots + \sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} (\Delta_1(\Delta_2 \dots (\Delta_n \lambda_{k_1,\dots,k_n}^{\theta}))) Y_{k_1,\dots,k_n}^{\theta}(f)_p < \infty,$$

then there exists a function $\varphi \in L_p^0(\mathbb{T}^n)$ with the Fourier series $\sigma(f, \lambda, \beta_1, \beta_2, \dots, \beta_n)$ and

$$\|\varphi\|_p \lesssim (\lambda_{1,1,\dots,1}^{\theta} \|f\|_p^{\theta} + \sum_{\nu_1=2}^{\infty} \mid \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-1},1,\dots,1}^{\theta} \mid Y_{2^{\nu_1}-1,0,\dots,0}^{\theta}(f)_p + \dots +$$

$$+ \sum_{\nu_n=1}^{\infty} \mid \Delta^{\varepsilon_n} \lambda_{1,\dots,2^{\nu_n-1}}^{\theta} \mid Y_{0,0,\dots,2^{\nu_n}-1}^{\theta}(f)_p + \sum_{\nu_2=1}^{\infty} \sum_{\nu_1=1}^{\infty} \mid \Delta^{\varepsilon_2} \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-1},2^{\nu_2-1},\dots,1}^{\theta} \mid Y_{2^{\nu_1}-1,2^{\nu_2}-1,\dots,0}^{\theta}(f)_p +$$
$$+ \dots + \sum_{\nu_n=1}^{\infty} \dots \sum_{\nu_1=1}^{\infty} \mid \Delta^{\varepsilon_n} \dots \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-1},2^{\nu_2-1},\dots,2^{\nu_n-1}}^{\theta} \mid Y_{2^{\nu_1}-1,2^{\nu_2-1},\dots,2^{\nu_n}-1}^{\theta}(f)_p)^{\frac{1}{\theta}}$$

And

$$Y_{2^{m_1}-1,2^{m_2}-1,\dots,2^{m_n}-1}(\varphi)_p \lesssim (\lambda_{2^{m_1-1},2^{m_2-1},\dots,2^{m_n-1}}^{\theta} Y_{2^{m_1}-1,2^{m_2}-1,\dots,2^{m_n}-1}(f)_p +$$

$$+ \sum_{\nu_1=m_1}^{k_1} |\Delta^{\varepsilon_1} \lambda_{2^{\nu_1-1},2^{m_2-1},\dots,2^{m_n-1}}^{\theta}| Y_{2^{\nu_1-1}-1,2^{m_2}-1,\dots,2^{m_n}-1}^{\theta}(f)_p + \dots +$$

$$+ \sum_{\nu_1=1}^{k_1} \sum_{\nu_2=1}^{k_2} |\Delta^{\varepsilon_1} \Delta^{\varepsilon_2} \lambda_{2^{\nu_1-1},2^{\nu_2-1},\dots,2^{m_n-1}}^{\theta}| Y_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,\dots,2^{m_n}-1}^{\theta}(f)_p$$

$$+ \dots + \sum_{\nu_1=1}^{k_1} \dots \sum_{\nu_n=1}^{k_n} |(\Delta^{\varepsilon_{n-1}}(\Delta^{\varepsilon_n} \dots (\Delta^{\varepsilon_n} \lambda_{2^{\nu_1},\dots,2^{\nu_n}}^{\theta})))| Y_{2^{\nu_1}-1,2^{\nu_2}-1,\dots,2^{\nu_{n-1}}-1}^{\theta}(f)_p)^{\frac{1}{\theta}},$$

where

$$\Delta^{\varepsilon_i} \lambda_{2^{k_1},\dots,2^{k_i},\dots,2^{k_n}} = \lambda_{2^{k_1},\dots,2^{k_{i+1}},\dots,2^{k_n}} - \lambda_{2^{k_1},\dots,2^{k_i},\dots,2^{k_n}}$$

Proof. Let series (3) be convergent, and $f \in L_p^0(T^n)$.We use the following inequality

$$\lambda^{\theta}_{2^{k_1-1},2^{k_2-1},\dots,2^{k_n-1}} \le \lambda^{\theta}_{1,1,\dots,1} + \sum_{m_1=2}^{k_1} |\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{m_1-2},1,\dots,1}| + \dots + \quad (4)$$

$$+ \dots + \sum_{m_n=2}^{k_n} |\Delta^{\varepsilon_n}\lambda^{\theta}_{1,1,\dots,2^{m_n-2}}| +$$

$$+ \dots + \sum_{m_1=2}^{k_1} \dots \sum_{m_n=2}^{k_n} |\Delta^{\varepsilon_1}(\Delta^{\varepsilon_2}\dots(\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{m_1-1},\dots,2^{m_n-1}}))|.$$

Using inequality (4), we get

$$I_1 = \left\{\int_0^{2\pi} \dots \int_0^{2\pi} \left[\sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \lambda^2_{2^{k_1-1},2^{k_2-1},\dots,2^{k_n-1}} D^2_{k_1,\dots,k_n}\right]^{\frac{p}{2}} dx_1,\dots,dx_n\right\}^{\frac{1}{p}} =$$

$$= \left\| \left[\sum_{k_1=1}^{\infty} \dots \sum_{k_n=1}^{\infty} \lambda^2_{2^{k_1-1},\dots,2^{k_n-1}} D^2_{k_1,\dots,k_n}\right]^{\frac{1}{2}} \right\|_p =$$

$$= \| \left[\lambda^2_{1,1,\dots,1} D^2_{1,1,\dots,1} + \sum_{k_1=2}^{\infty} \lambda^2_{2^{k_1-1},1,\dots,1} D^2_{k_1,1,\dots,1} + \sum_{k_2=2}^{\infty} \lambda^2_{1,2^{k_2-1,1,\dots,1}} D^2_{1,k_2,\dots,1} + \cdots \right.$$

$$+ \sum_{k_n=2}^{\infty} \lambda^2_{1,1,\dots,2^{k_n-1}} D^2_{1,1,\dots,2^{k_n-1}} + \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \lambda^2_{2^{k_1-1,2^{k_2-1,\dots,1}}{}_{D_1^2}} k^2_{k_1,k_2,\dots,1} + \cdots +$$

$$+ \sum_{k_1=2}^{\infty} \sum_{k_n=2}^{\infty} \lambda^2_{2^{k_1-1},1,\dots,2^{k_n-1}} D^2_{k_1,1,\dots,k_n} + \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \sum_{k_3=2}^{\infty} \lambda^2_{2^{k_1-1},2^{k_2-1},2^{k_3-1},1\dots,1} D^2_{k_1,k_2,k_3,\dots,1} +$$

$$\left. + \cdots + \sum_{k_1=2}^{\infty} \dots \sum_{k_n=2}^{\infty} \lambda^2_{2^{k_1-1},2^{k_2-1},\dots,2^{k_n-1}} D^2_{k_1,k_2,\dots,k_n}\right]^{\frac{1}{2}} \|_p$$

$$\le \| \left(\lambda^2_{1,1,\dots,1} D^2_{1,1,\dots,1} + \sum_{k_1=2}^{\infty} D^2_{k_1,1,1,\dots,1} \left[\lambda^{\theta}_{1,1,\dots,1} + \sum_{v_1=2}^{k_1} \left|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{v_1-2},1,\dots,1}\right|\right]^{\frac{2}{\theta}} + \right.$$

$$+ \cdots + \sum_{k_n=2}^{\infty} D^2_{1,1,\dots,k_n} \left[\lambda^{\theta}_{1,1,\dots,1} + \sum_{v_n=2}^{k_n} \left|\Delta^{\varepsilon_n}\lambda^{\theta}_{1,1,\dots,2^{v_n-2}}\right|\right]^{\frac{2}{\theta}} +$$

$$+ \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} D^2_{k_1,k_2\dots,1} \left[\lambda^{\theta}_{1,1,\dots1} + \sum_{v_1=2}^{k_1} \left|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{v_1-2},\dots,1}\right| + \sum_{v_2=2}^{k_2} \left|\Delta^{\varepsilon_2}\lambda^{\theta}_{1,2^{v_1-2,1,\dots,1}}\right| + \right.$$

$$\left. + \sum_{v_1=2}^{k_1} \sum_{v_n=2}^{k_n} \left|\Delta^{\varepsilon_1}\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{v_1-2},1,\dots,2^{v_n-2}}\right|\right]^{\frac{2}{\theta}} +$$

$$+ \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \dots \sum_{k_n=2}^{\infty} D^2_{k_1,k_2,\dots,k_n} [\lambda^{\theta}_{1,\dots,1,1} + \sum_{v_1=2}^{k_1} |\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{v_1-2},1,\dots,1}| +$$

$$+ \sum_{v_2=2}^{k_2} |\Delta^{\varepsilon_2}\lambda^{\theta}_{1,2^{v_2-2},1,\dots,1}| + \cdots + \sum_{v_n=2}^{k_n} |\Delta^{\varepsilon_n}\lambda^{\theta}_{1,1\dots,2^{v_2-2}}| + \sum_{v_1=2}^{k_1} \sum_{v_2=2}^{k_2} |\Delta^{\varepsilon_1}\Delta^{\varepsilon_2}\lambda^{\theta}_{2^{v_1-2},2^{v_2-2},1,\dots,1}| +$$

$$+\dots+\sum_{\nu_{n-1}=2}^{k_{n-1}}\sum_{\nu_n=2}^{k_n}|\Delta^{\varepsilon_{n-1}}\Delta^{\varepsilon_n}\lambda^{\theta}_{1,\dots,2^{\nu_{n-1}-2},2^{\nu_n-2}}|+$$

$$+\dots+\sum_{\nu_1=2}^{k_1}\sum_{\nu_2=2}^{k_2}\dots\sum_{\nu_n=2}^{k_n}|\Delta^{\varepsilon_1}\Delta^{\varepsilon_2}\dots\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{\nu_n-2}}|]^{\frac{2}{\theta}}\|_p.$$

Then we open the bracket and collect the expression of similar indices

$$I_1\le\|(\lambda^2_{1,1,\dots,1}D^2_{1,1,\dots,1}+\lambda^2_{1,1,\dots,1}\sum_{k_1=2}^{\infty}D^2_{k_1,1,\dots,1}+\lambda^2_{1,1,\dots,1}\sum_{k_2=2}^{\infty}D^2_{1,k_2,\dots,1}+\dots$$

$$+\lambda^2_{1,1,\dots,1}\sum_{k_n=2}^{\infty}D^2_{1,1,\dots,k_n}+\lambda^2_{1,1,\dots,1}\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}D^2_{k_1,k_2,1,\dots,1}+\dots+\lambda^2_{1,1,\dots,1}\sum_{k_1=2}^{\infty}\sum_{k_n=2}^{\infty}D^2_{k_1,1,\dots,k_n}+\dots$$

$$+\lambda^2_{1,1,\dots,1}\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\sum_{k_3=2}^{\infty}D^2_{k_1,k_2,k_3,\dots 1}+\dots+\lambda^2_{1,1,\dots,1}\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\dots\sum_{k_n=2}^{\infty}D^2_{k_1,k_2,\dots,k_n}+$$

$$\sum_{k_1=2}^{\infty}D^2_{k_1,1,\dots,1}[\sum_{\nu_1=2}^{k_1}|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}|]^{\frac{2}{\theta}}+\sum_{k_2=2}^{\infty}D^2_{1,k_2,\dots,1}[\sum_{\nu_2=2}^{k_2}|\Delta^{\varepsilon_2}\lambda^{\theta}_{1,2^{\nu_2-2},\dots,1}|]^{\frac{2}{\theta}}+\dots+$$

$$\sum_{k_n=2}^{\infty}D^2_{1,1,\dots,k_n}[\sum_{\nu_n=2}^{k_n}|\Delta^{\varepsilon_n}\lambda^{\theta}_{1,1,\dots,2^{\nu_n-2}}|]^{\frac{2}{\theta}}+\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}D^2_{k_1,k_2,\dots,1}[\textstyle\sum_{\nu_1=2}^{k_1}|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},\dots,1}|+$$

$$+\sum_{\nu_2=2}^{k_2}|\Delta^{\varepsilon_2}\lambda^{\theta}_{1,2^{\nu_2-2},1,\dots,1}|+\sum_{\nu_1=2}^{k_1}\sum_{\nu_2=2}^{k_2}|\Delta^{\varepsilon_1}\Delta^{\varepsilon_2}\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},1\dots,1}|]^{\frac{2}{\theta}}+..$$

$$+\dots+\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\dots\sum_{k_n=2}^{\infty}D^2_{k_1,k_2,\dots,k_n}[\sum_{\nu_1=2}^{k_1}|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}|+\dots+\sum_{\nu_n=2}^{k_n}|\Delta^{\varepsilon_n}\lambda^{\theta}_{1,\dots,2^{\nu_n-2}}|+$$

$$+\dots\dots+\sum_{\nu_1=2}^{k_1}\dots\sum_{\nu_n=2}^{k_n}|\Delta^{\varepsilon_1}\dots\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{\nu_n-2}}|]^{\frac{2}{\theta}})^{\frac{1}{2}}\|_p.$$

Next, we collect using similar indexes for all λ

$$I_1\lesssim\lambda_{1,1,\dots,1}\|[\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\dots\sum_{k_n=1}^{\infty}D^2_{k_1,k_2,\dots,k_n}]^{\frac{1}{2}}\|_p+$$

$$+\|(\sum_{k_1=2}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\dots\sum_{k_n=1}^{\infty}D^2_{k_1,k_2,\dots,k_n}[\sum_{\nu_1=2}^{k_1}|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}|]^{\frac{2}{\theta}})^{\frac{1}{2}}\|_p+$$

$$+\|(\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\sum_{k_3=1}^{\infty}\dots\sum_{k_n=1}^{\infty}D^2_{k_1,k_2,\dots,k_n}[\sum_{\nu_1=2}^{k_1}\sum_{\nu_2=2}^{k_2}|\Delta^{\varepsilon_1}\Delta^{\varepsilon_2}\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},\dots,1}|]^{\frac{2}{\theta}})^{\frac{1}{2}}\|_p+\dots+$$

$$+\|(\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\dots\sum_{k_n=2}^{\infty}D^2_{k_1,k_2,\dots,k_n}[\sum_{\nu_1=2}^{k_1}\sum_{\nu_2=2}^{k_2}\dots\sum_{\nu_n=2}^{k_n}|\Delta^{\varepsilon_1}\Delta^{\varepsilon_2}\dots\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{\nu_n-2}}|]^{\frac{2}{\theta}})^{\frac{1}{2}}\|_p$$

$$=:J_0+J_{\varepsilon_1}+\dots J_{\varepsilon_n}+J_{\varepsilon_1,\varepsilon_2}+\dots+J_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}.$$

Let us estimate J_0. Applying Lemma 6, we have $J_0\le C\lambda_{1,1,\dots,1}\|f\|_p<\infty$.

Now let's estimate J_{ε_1}:

$$J_{\varepsilon_1}=\{\int_0^{2\pi}\cdots\int_0^{2\pi}[\sum_{k_1=2}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}D^2_{k_1,k_2,\dots,k_n}[\sum_{\nu_1=2}^{k_1}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid]^{\frac{2}{\theta}}]^{\frac{p}{2}}dx_1,\dots,dx_n\}^{\frac{1}{p}}.$$

Using the Minkowski inequality for $\frac{2}{\theta}\geq 1$ several times, we obtain

$$\sum_{k_n=1}^{\infty}\cdots\sum_{k_2=1}^{\infty}\sum_{k_1=2}^{\infty}D^2_{k_1,k_2,\dots,k_n}[\sum_{\nu_1=2}^{k_1}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid]^{\frac{2}{\theta}}=$$

$$=\sum_{k_1=2}^{\infty}\sum_{k_2=1}^{\infty}\cdots((\sum_{k_n=1}^{\infty}\left[\sum_{\nu_1=2}^{k_1}\left|D_{k_1,k_2,\dots,k_n}\mid^{\theta}\right|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\right|]^{\frac{2}{\theta}})^{\frac{\theta}{2}})^{\frac{2}{\theta}}\leq$$

$$\leq\cdots\leq\sum_{k_1=2}^{\infty}(\sum_{\nu_1=2}^{k_1}\left[\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\left|D_{k_1,k_2,\dots,k_n}\mid^{2}\right|\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid^{\frac{2}{\theta}}\right]^{\frac{\theta}{2}})^{\frac{2}{\theta}}\leq$$

$$\leq(\sum_{\nu_1=2}^{\infty}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid(\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\mid D_{k_1,k_2,\dots,k_n}\mid^2)^{\frac{\theta}{2}})^{\frac{2}{\theta}}.$$

Now we substitute the obtained inequalities into our main integral, yielding the following expression

$$J_{\varepsilon_1}\leq\{\int_0^{2\pi}\cdots\int_0^{2\pi}[\{\sum_{\nu_1=2}^{\infty}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid(\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\mid D_{k_1,k_2,k_3}\mid^2)^{\frac{\theta}{2}}\}^{\frac{2}{\theta}}]^{\frac{p}{2}}dx_1\dots dx_n\}^{\frac{1}{p}}$$

$$=(\{\int_0^{2\pi}\cdots\int_0^{2\pi}(\sum_{\nu_1=2}^{\infty}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid(\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\mid D_{k_1,k_2,\dots,k_n}\mid^2)^{\frac{\theta}{2}})^{\frac{p}{\theta}}dx_1\dots dx_n\}^{\frac{\theta}{p}})^{\frac{1}{\theta}}.$$

Next we use the Minkowski inequality for $\frac{p}{\theta}\geq 1$, Lemmas 5 and 6, we have

$$J_{\varepsilon_1}\leq(\sum_{\nu_1=2}^{\infty}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid\{\int_0^{2\pi}\cdots\int_0^{2\pi}((\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\mid D_{k_1,k_2,\dots,k_n}\mid^2)^{\frac{\theta}{2}})^{\frac{p}{\theta}}dx_1\dots dx_n\}^{\frac{\theta}{p}})^{\frac{1}{\theta}}$$

$$=(\sum_{\nu_1=2}^{\infty}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-2},1,\dots,1}\mid\|(\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=1}^{\infty}\sum_{k_3=1}^{\infty}\mid D_{k_1,k_2,\dots,k_n}\mid^2)^{\frac{1}{2}}\|_p^{\theta})^{\frac{1}{\theta}}$$

$$\lesssim(\sum_{\nu_1=2}^{\infty}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-1},1,\dots,1}\mid Y^{\theta}_{2^{\nu_1}-1,0,\dots,0}(f)_p)^{\frac{1}{\theta}}.$$

Thus we get

$$J_{\varepsilon_1}\lesssim\left(\sum_{\nu_1=2}^{\infty}\mid\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-1},1,\dots,1}\mid\ Y^{\theta}_{2^{\nu_1-1},0,\dots,0}(f)_p\right)^{\frac{1}{\theta}}.$$

From (3) it follows that $J_{\varepsilon_1}<\infty$. Then $J_{\varepsilon_2},\dots,J_{\varepsilon_n}$ can be estimated similarly to J_{ε_1} and we have

$$J_{\varepsilon_2}\lesssim(\sum_{\nu_2=1}^{\infty}\mid\Delta^{\varepsilon_2}\lambda^{\theta}_{1,2^{\nu_2-1},\dots,1}\mid Y^{\theta}_{0,2^{\nu_2}-1,\dots,0}(f)_p)^{\frac{1}{\theta}},\dots,$$

$$J_{\varepsilon_n} \lesssim (\sum_{\nu_n=1}^{\infty} | \Delta^{\varepsilon_n} \lambda_{1,1,\dots,2^{\nu_2-1}}^{\theta} | Y_{0,0,\dots,2^{\nu_n-1}}^{\theta}(f)_p)^{\frac{1}{\theta}}.$$

To estimate $J_{\varepsilon_1,\varepsilon_2}$, we first obtain an upper bound for the following sum. Applying Lemma 3 for $\frac{2}{\theta} \geq 1$, we obtain

$$\sum_{k_1=1}^{\infty} \sum_{k_2=2}^{\infty} \cdots \sum_{k_n=2}^{\infty} D_{k_1,k_2,\dots,k_n}^2 [\sum_{\nu_1=2}^{k_1} \sum_{\nu_2=2}^{k_2} | \Delta^{\varepsilon_1} \Delta^{\varepsilon_2} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,1}^{\theta} |]^{2/\theta}$$

$$\leq \sum_{k_2=2}^{\infty} \cdots \sum_{k_n=1}^{\infty} (\sum_{\nu_1=2}^{\infty} \{\sum_{k_1=\nu_1}^{\infty} \left[\sum_{\nu_2=2}^{k_2} \left| D_{k_1,k_2,\dots,k_n} |^{\theta} \right| \Delta^{\varepsilon_2} \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,1}^{\theta} \right|]^{\frac{2}{\theta}} \}^{\frac{\theta}{2}})^{\frac{2}{\theta}}$$

$$\leq (\sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{\infty} | \Delta^{\varepsilon_2} \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,1}^{\theta} | (\sum_{k_1=\nu_1}^{\infty} \sum_{k_2=\nu_2}^{\infty} \cdots \sum_{k_n=1}^{\infty} | D_{k_1,k_2,\dots,k_n} |^2)^{\frac{\theta}{2}})^{\frac{2}{\theta}}$$

Hence, employing the inequalities derived above, let us now estimate $J_{\varepsilon_1,\varepsilon_2}$.
Applying Lemma 4, $\frac{p}{\theta} \geq 1$ it follows that

$$J_{\varepsilon_1,\varepsilon_2} = \{\int_0^{2\pi} \cdots \int_0^{2\pi} [\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \cdots \sum_{k_n=1}^{\infty} D_{k_1,k_2,\dots,k_n}^2 \times$$
$$\times [\sum_{\nu_1=2}^{k_1} \sum_{\nu_2=2}^{k_2} | \Delta^{\varepsilon_1} \Delta^{\varepsilon_2} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,1}^{\theta} |]^{\frac{2}{\theta}}]^{\frac{p}{2}} dx_1, dx_2, dx_3\}^{\frac{1}{p}}$$
$$\leq \{\int_0^{2\pi} \cdots \int_0^{2\pi} [\sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{\infty} | \Delta^{\varepsilon_2} \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,1}^{\theta} | \times$$
$$\times (\sum_{k_1=\nu_1}^{\infty} \sum_{k_2=\nu_2}^{\infty} \cdots \sum_{k_n=1}^{\infty} | D_{k_1,k_2,\dots,k_n} |^2)^{\frac{\theta}{2}})^{\frac{2}{\theta}}]^{\frac{p}{2}} dx_1, dx_2, \dots, dx_n\}^{\frac{1}{p}}$$
$$\leq (\sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{\infty} | \Delta^{\varepsilon_2} \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,1}^{\theta} | \times$$
$$\times \{\int_0^{2\pi} \cdots \int_0^{2\pi} [\sum_{k_1=\nu_1}^{\infty} \sum_{k_2=\nu_2}^{\infty} \cdots \sum_{k_n=1}^{\infty} | D_{k_1,k_2,\dots,k_n} |^2]^{\frac{p}{\theta}} dx_1, dx_2, \dots, dx_n\}^{\frac{\theta}{p}})^{\frac{1}{\theta}}$$

By Lemmas 5 and 6, we get

$$J_{\varepsilon_1,\varepsilon_2} \lesssim (\sum_{\nu_2=1}^{\infty} \sum_{\nu_1=1}^{\infty} | \Delta^{\varepsilon_2} \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-1},2^{\nu_2-1},\dots,1}^{\theta} | Y_{2^{\nu_1}-1,2^{\nu_2}-1,\dots,0}^{\theta}(f)_p)^{\frac{1}{\theta}}.$$

From (3) it follows that $J_{\varepsilon_1,\varepsilon_2} < \infty$. $J_{\varepsilon_1,\varepsilon_3}, \dots, J_{\varepsilon_{n-1},\varepsilon_n}$ can be estimated similarly to $J_{\varepsilon_1,\varepsilon_2}$ and we have

$$J_{\varepsilon_1,\varepsilon_3} \lesssim (\sum_{\nu_3=1}^{\infty} \sum_{\nu_1=1}^{\infty} | \Delta^{\varepsilon_3} \Delta^{\varepsilon_1} \lambda_{2^{\nu_1-1},1,2^{\nu_3-1},\dots,1}^{\theta} | Y_{2^{\nu_1}-1,0,2^{\nu_3}-1,\dots,0}^{\theta}(f)_p)^{\frac{1}{\theta}}, \dots,$$

$$J_{\varepsilon_{n-1},\varepsilon_n} \lesssim (\sum_{\nu_{n-1}=1}^{\infty} \sum_{\nu_n=1}^{\infty} | \Delta^{\varepsilon_n} \Delta^{\varepsilon_{n-1}} \lambda_{1,1,\dots,2^{\nu_{n-1}-1},2^{\nu_n-1}}^{\theta} | Y_{0,\dots,2^{\nu_{n-1}}-1,2^{\nu_n}-1}^{\theta}(f)_p)^{\frac{1}{\theta}}.$$

To estimate $J_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}$, we first obtain an upper bound for the following sum.

Applying Lemmas 3 and 4 for $\frac{2}{\theta} \geq 1$, we derive

$$\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \cdots \sum_{k_n=2}^{\infty} D_{k_1,k_2,\dots,k_n}^2 [\sum_{\nu_1=2}^{k_1} \sum_{\nu_2=2}^{k_2} \cdots \sum_{\nu_n=2}^{k_n} |\prod_{i=1}^{n} \Delta^{\varepsilon_i} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{\nu_n-2}}^{\theta} |]^{\frac{2}{\theta}} \leq$$

$$\leq (\sum_{\nu_n=2}^{\infty} \sum_{\nu_{n-1}=2}^{\infty} \cdots \sum_{\nu_1=2}^{\infty} [\sum_{k_1=\nu_1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} [| D_{k_1,k_2,\dots,k_n} |^{\theta}$$

$$|\prod_{i=1}^{n} \Delta^{\varepsilon_i} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{\nu_n-2}}^{\theta} |]^{\frac{2}{\theta}}]^{\frac{\theta}{2}}\}^{\frac{2}{\theta}} = (\sum_{\nu_n=2}^{\infty} \sum_{\nu_{n-1}=2}^{\infty} \cdots \sum_{\nu_1=2}^{\infty}$$

$$|\prod_{i=1}^{n} \Delta^{\varepsilon_i} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{\nu_n-2}}^{\theta} | [\sum_{k_1=\nu_1}^{\infty} \cdots \sum_{k_n=\nu_n}^{\infty} | D_{k_1,k_2,\dots,k_n} |^2]^{\frac{\theta}{2}})^{\frac{2}{\theta}}$$

Therefore, from Lemma 4 ($\frac{p}{\theta} \geq 1$) it follows that

$$J_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n} = \{\int_0^{2\pi} \dots \int_0^{2\pi} [\sum_{k_1=2}^{\infty} \dots \sum_{k_n=2}^{\infty} D_{k_1,\dots,k_n}^2 [\sum_{\nu_1=2}^{\infty} \cdots$$

$$\cdots \sum_{\nu_n=2}^{\infty} |\prod_{i=1}^{n} \Delta^{\varepsilon_i} \lambda_{2^{\nu_1-2},\dots,2^{\nu_n-2}}^{\theta} |]^{\frac{2}{\theta}}]^{\frac{p}{2}} dx_1 \dots dx_n\}^{\frac{1}{p}} \leq$$

$$\leq (\sum_{\nu_n=2}^{\infty} \sum_{\nu_1=2}^{\infty} |\prod_{i=1}^{n} \Delta^{\varepsilon_i} \lambda_{2^{\nu_1-2},\dots,2^{\nu_n-2}}^{\theta} | \{\int_0^{2\pi} \cdots \int_0^{2\pi} [(\sum_{k_1=\nu_1}^{\infty} \cdots$$

$$\cdots \sum_{k_n=\nu_n}^{\infty} | D_{k_1,\dots,k_n} |^2]^{\frac{p}{\theta}} dx_1 \dots dx_n\}^{\frac{\theta}{p}})^{\frac{1}{\theta}}$$

Taking into account Lemmas 5 and 6, we obtain

$$J_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n} \lesssim \sum_{\upsilon_n=2}^{\infty} \dots \sum_{\nu_1=2}^{\infty} |\prod_{i=1}^{n} \Delta^{\varepsilon_i} \lambda_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{\nu_n-2}}^{\theta} \left| Y_{2^{\nu_1-1},2^{\nu_2-1},\dots,2^{\nu_n-1}}^{\theta} (f)_p \right)^{\frac{1}{\theta}}.$$

From (3), we get $J_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n} < \infty$. By collecting estimates $J_0, J_{\varepsilon_1}, J_{\varepsilon_2}, \dots, J_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}$, we obtain $I_1 < \infty$. Hence, by Lemma 6 there exists a function $g(x_1, x_2, \dots, x_n) \in L_p^0$ with the Fourier series

$$\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \lambda_{2^{k_1-1},2^{k_2-1},\dots,2^{k_n-1}} D_{k_1,k_2,\dots,k_n}, \tag{5}$$

and

$$\|g\|_p \leq C(p) I_1. \tag{6}$$

We rewrite series (5) in the form of

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\gamma_{k_1,k_2,\dots,k_n}A_{k_1,k_2,\dots,k_n}(x_1,x_2,\dots,x_n),$$

Where

$\gamma_{1,1,\dots,1}=\lambda_{1,1,\dots,1}$, $\gamma_{\nu_1,1,\dots,1}=\lambda_{2^{k_1-1},1,\dots1}$ for $2^{k_1-1}\le\nu_1\le 2^{k_1}-1$ $(k_1=2,3\dots)$,

$\gamma_{1,\nu_2,\dots,1}=\lambda_{1,2^{n_2-1},\dots,1}$ for $2^{k_2-1}\le\nu_2\le 2^{n_2}-1, (k_2=2,3\dots),\dots,$

$\gamma_{\nu_1,\nu_2,\dots,1}=\lambda_{2^{k_1-1},2^{k_2-1},\dots,1}$ for $2^{k_1-1}\le\nu_1\le 2^{k_1}-1, 2^{k_2-1}\le\nu_2\le 2^{k_2}-1, (k_1,k_2=2,3\dots)$,

$\gamma_{\nu_1,\nu_2,\dots,\nu_n}=\lambda_{2^{k_1-1},2^{k_2-1},\dots2^{k_n-1}}$ for $2^{k_1-1}\le\nu_1\le 2^{k_1}-1, 2^{k_2-1}\le\nu_2\le 2^{k_2}-1,\dots,$

$2^{k_n-1}\le\nu_n\le 2^{k_n}-1(k_1,k_2,\dots,k_n=2,3\dots)$.

Let us consider the following series

Where

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\lambda_{k_1,k_2,\dots,k_n}A_{k_1,k_2,\dots,k_n}(x_1,x_2,\dots,x_n)$$
$$=\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\dots\sum_{k_n=1}^{\infty}\gamma_{k_1,k_2,\dots,k_3}\Lambda_{k_1,k_2,\dots,k_n}A_{k_1,k_2,\dots,k_n}(x_1,x_2,\dots,x_n) \quad (7)$$

$\Lambda_{1,1,\dots,1}=1,$

$\Lambda_{\nu_1,1,\dots,1}=\frac{\lambda_{\nu_1,1,\dots,1}}{\gamma_{\nu_1,1,\dots,1}}=\frac{\lambda_{\nu_1,1,\dots,1}}{\lambda_{2^{k_1-1},1,\dots,1}}$ for $2^{k_1-1}\le\nu_1\le 2^{k_1}-1,$ $(k_1=2,3\dots)$,

$\Lambda_{1,1,\dots,\nu_n}=\frac{\lambda_{1,1,\dots,\nu_n}}{\gamma_{1,1,\dots,\nu_n}}=\frac{\lambda_{1,1,\dots,\nu_n}}{\lambda_{1,1,\dots,2^{k_n-1}}}$ for $2^{k_{3n}-1}\le\nu_n\le 2^{k_n}-1,$ $(k_n=2,3\dots)$,

$\Lambda_{\nu_1,\nu_2,\dots,1}=\frac{\lambda_{\nu_1,\nu_2,\dots,1}}{\gamma_{\nu_1,\nu_2,\dots,1}}=\frac{\lambda_{\nu_1,\nu_2,\dots,1}}{\lambda_{2^{k_1-1},2^{k_2-1},\dots,1}}$ for $2^{k_1-1}\le\nu_1\le 2^{k_1}-1$,
$2^{k_2-1}\le\nu_2\le 2^{k_2}-1, (k_1,k_2=2,3\dots)$

$\Lambda_{\nu_1,\nu_2,\dots,\nu_n}=\frac{\lambda_{\nu_1,\nu_2,\dots,\nu_3}}{\gamma_{\nu_1,\nu_2,\dots,\nu_n}}=\frac{\lambda_{\nu_1,\nu_2,\dots,\nu_n}}{\lambda_{2^{k_1-1},2^{k_2-1},\dots,2^{k_n-1}}}$ for $2^{k_i-1}\le\nu_i\le 2^{k_i}-1$,

$i=1,2,\dots,n, (k_2,\dots,k_n=2,3\dots)$.

Since the sequence $\{\lambda_{k_1,k_2,\dots,k_n}\}=\lambda\in GM^n$, the sequence $\{\Lambda_{k_1,k_2,\dots,k_n}\}_{k_1=1,\dots,k_n=1}^{\infty,\dots,\infty}$ satisfies the conditions of Marcinkiewicz multiplier theorem, then series (7) is the Fourier series of a function $\varphi(x_1,x_2,x_2)\in L_p$ and $\|\varphi\|_p\le C(\rho,\lambda)\|g\|_p$.

Applying (6) and estimates $J_0, J_{\varepsilon_1}, J_{\varepsilon_2},\dots, J_{\varepsilon_n}, J_{\varepsilon_1,\varepsilon_2},\dots, J_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}$, we have

$$\|\varphi\|_p \lesssim (\lambda^{\theta}_{1,1,\ldots,1}\|f\|^{\theta}_p + \sum_{\nu_1=2}^{\infty} | \Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-1},1,\ldots,1} | Y^{\theta}_{2^{\nu_1}-1,0,\ldots,0}(f)_p +$$

$$+\sum_{\nu_2=1}^{\infty} | \Delta^{\varepsilon_2}\lambda^{\theta}_{1,2^{\nu_2-1},\ldots,1} | Y^{\theta}_{0,2^{\nu_2}-1,\ldots,0}(f)_p + \ldots + \sum_{\nu_n=1}^{\infty} | \Delta^{\varepsilon_n}\lambda^{\theta}_{1,\ldots,2^{\nu_n-1}} | Y^{\theta}_{0,0,\ldots,2^{\nu_n}-1}(f)_p +$$

$$+\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty} | \Delta^{\varepsilon_2}\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},,1\ldots1} | Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1,\ldots,0}(f)_p +$$

$$+\ldots+\sum_{\nu_n=1}^{\infty}\ldots\sum_{\nu_1=1}^{\infty} | \Delta^{\varepsilon_n}\ldots\Delta^{\varepsilon_1}\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},\ldots,2^{\nu_n-1}} | Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2-1},\ldots,2^{\nu_n}-1}(f)_p).$$

Further, we estimate $Y_{2^{k_1}-1},\ldots,_{2^{k_n}-1}(\varphi)_p$. Using Lemma 5, we derive

$$Y_{2^{k_1}-1},\ldots,_{2^{k_n}-1}(\varphi)_p \leq \|\varphi - U_{2^{k_1}-1},\ldots,_{2^{k_n}-1}(\varphi)\|_p$$
$$\leq \sum_{\nu_n=k_n}^{\infty}\ldots\sum_{\nu_1=k_1}^{\infty}\|\sum_{s_n=2^{\nu_n}+1}^{2^{\nu_n+1}}\ldots\sum_{s_1=2^{\nu_1}+1}^{2^{\nu_1+1}} D_s(\varphi)\|_p.$$

We consider the series (7)

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\lambda_{k_1,k_2,\ldots,k_n}A^*_{k_1,k_2,\ldots,k_n}(x_1,x_2\ldots,x_n) =$$

$$= \sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\gamma_{k_1,k_2,\ldots,k_n}\Lambda_{k_1,k_2,\ldots,k_n}A^*_{k_1,k_2,\ldots,k_n}(x_1,x_2,\ldots,x_n),$$

where $A^*_{k_1,k_2,\ldots,k_n}(x_1,x_2,\ldots,x_n) = 0$, if $k_1 \leq 2^{m_1}-1$ and $k_2 \leq 2^{m_2}-1$,..., $k_n \leq 2^{m_n}-1$, also in other cases $A^*_{k_1,k_2,\ldots,k_n}(x_1,x_2,\ldots,x_n) = A_{k_1,k_2,\ldots,k_n}(x_1,x_2,\ldots,x_n)$. Since the sequence $\{\Lambda_{k_1,k_2,\ldots,k_n}\}$ satisfies the conditions of Marcinkiewicz multiplier theorem, then

$$\|\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\cdots\sum_{k_n=1}^{\infty}\lambda_{k_1,k_2,\ldots,k_n}A^*_{k_1,k_2,\ldots,k_n}(x_1,x_2,\ldots,x_n)\|_p \leq$$

$$\leq C\|\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\ldots\sum_{k_n=1}^{\infty}\lambda_{2^{n_1-1},2^{n_2-1},\ldots,2^{k_n-1}}D^*_{k_1,k_2,\ldots,k_n}\|_p$$

where $D^*_{k_1,k_2,\ldots,k_n} = 0$, if $k_1 \leq m_1$, $k_2 \leq m_2$,... , $k_n \leq m_n$ and in other cases $D^*_{k_1,k_2,\ldots,k_n} = D_{k_1,k_2,\ldots,k_n}$.

By Lemma 6, we get

$$Y_{2^{m_1-1},2^{m_2-1},\ldots,2^{m_n-1}}(\varphi)_p \lesssim \{\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi}[$$

$$[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\ldots\sum_{k_n=m_n+1}^{\infty}\lambda^2_{2^{k_1-1},2^{k_2-1},\ldots,2^{k_n-1}}D^2_{k_1,k_2,\ldots,k_n}]^{\frac{p}{2}}dx_1dx_2\ldots dx_n\}^{\frac{1}{p}}. \quad (8)$$

It's easy to see

$$\lambda^{\theta}_{2^{k_1-1},2^{k_2-1},\ldots,2^{k_n-1}} \leq \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},\ldots,2^{m_n-1}} +$$

$$+\sum_{\nu_1=m_1+1}^{k_1}\left|\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},\dots,2^{m_n-1}}\right|+$$

$$+\sum_{\nu_{n-1}=m_n+1}^{k_n-1}\sum_{\nu_n=m_n+1}^{k_n}|\lambda^{\theta}_{2^{m_n-1},\dots,2^{\nu_{n-1}-1},2^{\nu_n-1}}-\lambda^{\theta}_{2^{m_n-1},\dots,2^{\nu_{n-1}-2},2^{\nu_n-1}}-\lambda^{\theta}_{2^{m_n-1},\dots,2^{\nu_{n-1}-1},2^{\nu_n-2}}+$$

$$+\lambda^{\theta}_{2^{m_n-1},\dots,2^{\nu_{n-1}-2},2^{\nu_n-2}}|+\sum_{\nu_1=m_1+1}^{k_1}\dots\sum_{\nu_n=m_n+1}^{k_n}|\Delta^{\varepsilon_1}(\Delta^{\varepsilon_2}\dots(\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-1},\dots,2^{\nu_n-1}})))|.$$

Substituting this estimate into (2), we obtain

$$Y_{2^{m_1-1},2^{m_2-1},\dots,2^{m_n-1}}(\varphi)_p\le$$

$$\le\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}(\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},\dots,2^{m_n-1}}+$$

$$+\sum_{\nu_1=m_1+1}^{k_1}|\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},\dots,2^{m_n-1}}|+$$

$$+\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},\dots,2^{m_n-1}}-$$

$$-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},\dots,2^{m_n-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{m_n-1}}|+$$

$$+..+\sum_{\nu_1=m_1+1}^{k_1}\dots\sum_{\nu_n=m_n+1}^{k_n}|\Delta^{\varepsilon_1}(\Delta^{\varepsilon_2}\dots(\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-1},\dots,2^{\nu_n-1}})))|)^{\frac{2}{\theta}}D^2_{k_1,k_2,\dots,k_n}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{1}{p}}$$

$$\lesssim(\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},\dots,2^{m_n-1}}\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots$$

$$\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{1}{p}}$$

$$+\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}$$

$$(\sum_{\nu_1=m_1+1}^{k_1}|\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},\dots,2^{m_n-1}}|)^{\frac{2}{\theta}}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{1}{p}}+$$

$$+\{\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},\dots,2^{m_n-1}}$$

$$-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2},\dots,2^{m_n-1}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2},\dots,2^{m_n-1}}|)^{\frac{2}{\theta}}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{1}{p}}+$$

$$+\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\sum_{k_3=m_3+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}$$

$$(\sum_{\nu_1=m_1+1}^{k_1}\dots\sum_{\nu_n=m_n+1}^{k_n}|\Delta^{\varepsilon_1}(\Delta^{\varepsilon_2}\dots(\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-1},\dots,2^{\nu_n-1}})))|)^{\frac{2}{\theta}}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{1}{p}}$$

$$=:L_0+L_{\varepsilon_1}+\dots+L_{\varepsilon_n}+L_{\varepsilon_1,\varepsilon_2}+\dots+L_{\varepsilon_{n-1},\varepsilon_n}+\dots+L_{\varepsilon_1,\dots\varepsilon_n}.$$

We estimate L_0 as J_0, to get

$$L_0 \le \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},\dots,2^{m_n-1}}\{\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}]^{\frac{p}{2}}dx_1,dx_2,\dots,dx_n\}^{\frac{1}{p}}$$
$$\lesssim \lambda^{\theta}_{2^{m_1-1},2^{m_2-1},\dots,2^{m_n-1}}Y_{2^{m_1}-1,2^{m_2}-1,\dots,2^{m_n}-1}(f)_p.$$

Using an estimation method similar to the J_{ε_1} method, we obtain the estimate L_{ε_1}

$$L_{\varepsilon_1}=\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}$$
$$(\sum_{\nu_1=m_1+1}^{k_1}|\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},\dots,2^{m_n-1}}|)^{\frac{2}{\theta}}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{1}{p}}\lesssim$$
$$\lesssim(\sum_{\nu_1=m_1+1}^{k_1}|\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},\dots,2^{m_n-1}}|$$
$$\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{\theta}{p}})^{\frac{1}{\theta}}$$
$$\lesssim(\sum_{\nu_1=m_1+1}^{k_1}|\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},\dots,2^{m_n-1}}-\lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1},\dots,2^{m_n-1}}|\,Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,\dots,2^{m_n}-1}(f)_p)^{\frac{1}{\theta}}.$$

In the case where $L_{\varepsilon_2}, L_{\varepsilon_3}$ and so on L_{ε_n} we estimate as L_{ε_1}.

We estimate $L_{\varepsilon_{n-1},\varepsilon_n}$ as follows

$$L_{\varepsilon_{n-1},\varepsilon_n}=\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}\times$$
$$\times(\sum_{\nu_{n-1}=m_n+1}^{k_n-1}\sum_{\nu_n=m_n+1}^{k_n}|\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-1},2^{\nu_n-1}}-\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-2},2^{\nu_n-1}}-\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-1},2^{\nu_n-2}}$$
$$+\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-2},2^{\nu_n-2}}|)^{\frac{2}{\theta}}]^{\frac{p}{2}}dx_1dx_2\dots dx_n\}^{\frac{1}{p}}\lesssim$$

$$\lesssim(\sum_{\nu_{n-1}=m_n+1}^{k_n-1}\sum_{\nu_n=m_n+1}^{k_n}|\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-1},2^{\nu_n-1}}-\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-2},2^{\nu_n-1}}-\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-1},2^{\nu_n-2}}$$
$$+\lambda^{\theta}_{2^{m_1-1},\dots,2^{\nu_{n-1}-2},2^{\nu_n-2}}|Y^{\theta}_{2^{m_1}-1,\dots,2^{\nu_{n-1}-1}-1,2^{\nu_{n-1}}-1,}(f)_p)^{\frac{1}{\theta}}.$$

Similarly, we obtain estimates for $L_{\varepsilon_1,\dots\varepsilon_n}$

$$L_{\varepsilon_1,\dots\varepsilon_n}=\{\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\dots\sum_{k_n=m_n+1}^{\infty}D^2_{k_1,k_2,\dots,k_n}\lesssim$$

$$\lesssim (\sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_n=m_n+1}^{k_n} |\Delta^{\varepsilon_1}(\Delta^{\varepsilon_2}\ldots(\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-1},\ldots,2^{\nu_n-1}})))|)^{\frac{2}{\theta}}]^{\frac{p}{2}} dx_1 dx_2 \ldots dx_n\}^{\frac{1}{p}} \lesssim$$

$$\lesssim (\sum_{\nu_1=m_1+1}^{k_1} \ldots \sum_{\nu_n=m_n+1}^{k_n} |\Delta^{\varepsilon_1}(\Delta^{\varepsilon_2}\ldots(\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1-1},\ldots,2^{\nu_n-1}})))| Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1,\ldots,2^{\nu_n}-1}(f)_p)^{\frac{1}{\theta}}.$$

Now we collect estimates $L_0, L_{\varepsilon_1}, \ldots, L_{\varepsilon_n}, L_{\varepsilon_1,\varepsilon_2}, \ldots, L_{\varepsilon_1,\ldots\varepsilon_n}$, we derive

$$Y_{2^{m_1}-1, 2^{m_2}-1, \ldots, 2^{m_n}-1}(\varphi)_p \lesssim (\lambda^{\theta}_{2^{m_1-1},2^{m_2-1},\ldots,2^{m_n-1}} Y_{2^{m_1}-1, 2^{m_2}-1, \ldots, 2^{m_n}-1}(f)_p$$
$$+\sum_{\nu_1=m_1}^{k_1} |\lambda^{\theta}_{2^{\nu_1},2^{m_2-1},\ldots,2^{m_n-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1},\ldots,2^{m_n-1}}|\, Y^{\theta}_{2^{\nu_1-1}-1,2^{m_2}-1,\ldots,2^{m_n}-1}(f)_p + \ldots +$$
$$+\sum_{\nu_1=1}^{k_1} \sum_{\nu_2=1}^{k_2} |\lambda^{\theta}_{2^{\nu_1},2^{\nu_2},\ldots,2^{m_n-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2},\ldots,2^{m_n-1}}$$
$$-\lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1},\ldots,2^{m_n-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1},\ldots,2^{m_n-1}}|\, Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1,\ldots,2^{m_n}-1}(f)_p$$
$$+\ldots+\sum_{\nu_{n-1}=1}^{k_n-1} \sum_{\nu_n=1}^{k_n} |\lambda^{\theta}_{2^{m_1-1},\ldots,2^{\nu_{n-1}},2^{\nu_n}} - \lambda^{\theta}_{2^{m_1-1},\ldots,2^{\nu_{n-1}-1},2^{\nu_n}} - \lambda^{\theta}_{2^{m_1-1},\ldots,2^{\nu_{n-1}},2^{\nu_n-1}}$$
$$+\lambda^{\theta}_{2^{m_1-1},\ldots,2^{\nu_{n-1}-1},2^{\nu_n-1}}| Y^{\theta}_{2^{m_1}-1,\ldots,2^{\nu_{n-1}-1}-1,2^{\nu_n-1}-1}(f)_p$$
$$+\sum_{\nu_1=1}^{k_1} \ldots \sum_{\nu_n=1}^{k_n} |\Delta^{\varepsilon_1}(\Delta^{\varepsilon_2}\ldots(\Delta^{\varepsilon_n}\lambda^{\theta}_{2^{\nu_1},\ldots,2^{\nu_n}})))| Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1,\ldots,2^{\nu_n-1}-1}(f)_p)^{\frac{1}{\theta}}.$$

The proof of Theorem is complete.

Discussion

In this paper, estimates of the best angular approximations of functions are obtained, generalizing the previously known results of Potapov [1], and extending them to the case of multidimensional generalized monotone sequences and generalized derivatives in the Liouville-Weyl sense. In [5], the one-dimensional case is considered and upper approximation estimates are established based on the characteristics of functions, such as their smoothness and singularities. In [8], in turn, these results are extended to the two-dimensional case, which significantly expands the scope of their application. Based on these theorems, a methodology for multidimensional analysis of angular approximations is developed, taking into account not only the main parameters of functions, but also their complex structure in multidimensional spaces. The results obtained show that approximation of functions in the multidimensional case requires more complex estimates associated with additional parameters, such as the relationships between variables and the features of their interaction. Thus, the proposed methods and estimates not only generalize the known results, but also provide a new approach to the study of angular approximations in the multidimensional case, which is an important contribution to the development of approximation theory.

Conclusion

In this paper, optimal angular approximations of multivariable functions were investigated, which allowed for the derivation of upper bounds for these approximations. The main results showed that the quality of function approximation significantly depended on the characteristics of the functions, including their smoothness, dimensionality, and presence of singularities. The developed methods and obtained estimates provided a powerful tool for analyzing multivariate functions, offering both theoretical significance. Thus, this work made a significant contribution to the theory of multivariate approximations, opening new opportunities for solving problems in mathematics and related fields.

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