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A CLASS OF INVERSE PROBLEMS FOR THE HEAT EQUATION WITH INVOLUTIVE PERTURBATION

Abstract

Inverse problems for the equation of deflected thermal conductivity with involution are one of the most relevant research topics in the field of mathematical physics. This study is devoted to the study of solutions of deviating equations characterizing the process of thermal conductivity, and the development of methods for solving inverse problems, taking into account their involutional properties. Such tasks are widely used in practical applications such as studies of thermal properties of materials, problems of reverse distribution, and engineering tasks for managing thermal processes. The equations of deflected thermal conductivity with involution are a general modified version of the problem of thermal conductivity, which allows us to more accurately describe various physical processes. In such equations, higher sequences of time derivatives or additional involutive terms are introduced, which complicates the model, but brings it closer to real processes. The theory of inverse problems includes important questions in the search for solutions to the equations of thermal conductivity. By determining unknown coefficients, initial or boundary conditions based on actual data, these tasks allow a deeper understanding of thermal processes. The specificity of deviating equations is due to the need to preserve the stability and uniqueness of their solutions. The purpose of this work is to study inverse problems for equations of deflected thermal conductivity with involution, and to develop analytical methods for their solution. The paper considers the issues of setting inverse problems, studying the conditions for their correct formulation, proving loneliness and stability of solutions. In addition, effective methods for solving problems are proposed. The novelty of the work lies in the presentation of a new formulation of inverse problems for equations with involution and the study of their analytical solutions. These models allow us to describe specific physical phenomena, such as the thermal conductivity of complex materials or changes caused by external factors. In addition, the results of the study contribute to improving the accuracy of the model in solving many engineering and scientific problems. The results of the work make a significant contribution to the development of the theory of inverse problems, as well as to the construction of new mathematical models of thermal conductivity processes. The results of the research can be used in scientific research, engineering reports and optimization of technological processes. A class of inverse problems for the equation of deflected thermal conductivity with involution is considered using four different boundary conditions. The solutions were obtained in the form of series classification using sets orthogonal to each report. The completeness of the solutions received was also discussed.

Keywords: Inverse Problems, Heat Equation, Involution Perturbation, Boundary Condition, Equation.

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КЕРІ ЕСЕПТЕР КЛАСЫ ҮШІН ИНВОЛЮЦИЯЛЫҚ АУЫТҚУЫ БАР ЖЫЛУ ӨТКІЗГІШТІК ТЕҢДЕУІ

Аңдатпа

Инволюциясы бар ауытқыған жылу өткізгіштік теңдеуіне арналған кері есептер математикалық физика саласындағы өзекті зерттеу тақырыптарының бірі болып табылады. Бұл зерттеу жылу өткізгіштік процесін сипаттайтын ауытқыған теңдеулердің шешімдерін зерттеуге және олардың инволюциялық қасиеттерін ескере отырып кері есептерді шешу әдістемесін дамытуға арналған. Мұндай есептер практикалық қолданбаларда кеңінен кездеседі, мысалы, материалдардың жылу қасиеттерін зерттеу, кері тарату мәселелері, және жылулық процестерді басқарудағы инженерлік есептер. Инволюциясы бар ауытқыған жылу өткізгіштік теңдеулері – жылу өткізгіштік мәселесінің жалпы түрлендірілген нұсқасы, ол әртүрлі физикалық процестерді дәлірек сипаттауға мүмкіндік береді. Мұндай теңдеулерде уақыттық туындылардың жоғары реттілігі немесе қосымша инволютивті мүшелер енгізіледі, бұл модельді күрделендіреді, бірақ нақты процестерге жақындатады. Кері есептер теориясы жылу өткізгіштік теңдеулерінің шешімін табудағы маңызды мәселелерді қамтиды. Нақты деректер негізінде белгісіз коэффициенттерді, бастапқы немесе шеттік шарттарды анықтау арқылы бұл есептер жылулық процестерді тереңірек түсінуге мүмкіндік береді. Ауытқыған теңдеулердің ерекшелігі олардың шешімдерінің тұрақтылығы мен бірегейлігін сақтау қажеттілігінен туындайды. Бұл жұмыстың мақсаты – инволюциясы бар ауытқыған жылу өткізгіштік теңдеулері үшін кері есептерді зерттеу, оларды шешудің аналитикалық әдістерін дамыту. Жұмыста кері есептерді қою, олардың дұрыс қойылым шарттарын зерттеу, шешімдердің жалғыздығы мен орнықтылығын дәлелдеу мәселелері қарастырылады. Сонымен қатар, есептерді шешудің тиімді әдістері ұсынылады. Жұмыстың жаңалығы – инволюциясы бар теңдеулер үшін кері есептердің жаңа қойылымын ұсыну және олардың аналитикалық шешімдерін зерттеу. Бұл модельдер нақты физикалық құбылыстарды сипаттауға мүмкіндік береді, мысалы, күрделі материалдардың жылу өткізгіштігі немесе сыртқы факторлардың әсерінен болатын өзгерістер. Сонымен қатар, зерттеудің нәтижелері көптеген инженерлік және ғылыми есептерді шешу кезінде модельдің дәлдігін арттыруға ықпал етеді. Жұмыстың нәтижелері кері есептер теориясын дамытуға, сондай-ақ жылу өткізгіштік процестерінің жаңа математикалық модельдерін құруға елеулі үлес қосады. Зерттеу нәтижелері ғылыми-зерттеу жұмыстарында, инженерлік есептерде және технологиялық процестерді оңтайландыруда қолданылуы мүмкін. Инволюциясы бар ауытқыған жылу өткізгіштік теңдеуіне арналған кері есептер класы төрт түрлі шекаралық шарттарын қолдану арқылы қарастырылады. Шешімдер әрбір есепке сәйкес ортогонал болатын жиындарды қолдана отырып қатарға жіктеу түрінде алынды. Алынған шешімдердің жинақтылығы да талқыланды.

Түйін сөздер: кері есептер, жылуөткізгіштік теңдеуі, инволюциялық ауытқу, шеттік шарт, теңдеу.

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УРАВНЕНИЕ ТЕПЛОПРОВОДНОСТИ С ИНВОЛЮЦИОННЫМ ВОЗМУЩЕНИЕМ ДЛЯ КЛАССА ОБРАТНЫХ ЗАДАЧ

Аннотация

Обратные задачи для отклоняющегося уравнения теплопроводности с инволюцией являются одной из актуальных тем исследований в области математической физики. Данная работа посвящена исследованию решений уравнений отклонений, описывающих процесс теплопроводности, и разработке методики решения обратных задач с учетом их инволюционных свойств. Такие проблемы распространены в практических приложениях, таких как изучение термических свойств материалов, проблемы обратного распространения ошибки и инженерные проблемы управления термическими процессами. Отклоненные уравнения теплопроводности с инволюцией представляют собой общий модифицированный вариант задачи теплопроводности, позволяющий более точно описать различные физические процессы. В такие уравнения вводятся производные по времени более высокого порядка или дополнительные инволютивные члены, что усложняет модель, но приближает ее к реальным процессам. Теория обратных задач связана с важными проблемами решения уравнений теплопроводности. Путем определения неизвестных коэффициентов, начальных или граничных условий на основе реальных данных эти задачи позволяют глубже понять тепловые процессы. Своеобразие дифференциальных уравнений возникает из необходимости сохранения устойчивости и единственности их решений. Целью данной работы является исследование обратных задач для девиационных уравнений теплопроводности с инволюцией, разработка аналитических методов их решения. Работа посвящена проблемам постановки обратных задач, исследованию условий их корректной постановки, доказательству единственности и устойчивости решений. Кроме того, предлагаются эффективные методы решения проблем. Новизна работы заключается в представлении новой серии обратных задач для уравнений с инволюцией и исследовании их аналитических решений. Эти модели позволяют описывать реальные физические явления, например, теплопроводность сложных материалов или изменения, вызванные внешними факторами. Кроме того, результаты исследований способствуют повышению точности модели при решении многих инженерных и научных задач. Результаты работы вносят существенный вклад в развитие теории обратных задач, а также в создание новых математических моделей процессов теплопроводности. Результаты исследований могут быть использованы в научных исследованиях, инженерных отчетах и оптимизации технологических процессов. Рассмотрен класс обратных задач для девиаторного уравнения теплопроводности с инволюцией с использованием четырех различных граничных условий. Решения были получены в виде ранговой классификации с использованием наборов, ортогональных каждому отчету. Обсуждалась также последовательность решений.

Ключевые слова: обратные задачи, уравнение теплопроводности, отклонение инволюции, граничные условия, уравнение.

Main provisions

Equations involving an unknown function and its derivatives, taken generally at different values of the argument, are called nonlocal differential equations. Equations with an involutive shift of the argument, a special case of nonlocal differential equations [1]. The transformation S is called an involution if $\alpha^2(t)=\alpha(\alpha(t))=t.$ Differential equations with an involutive shift in the unknown function or its derivative are known to be model equations with a variable shift of the argument. In general, such equations can be classified as functional-differential equations. This article considers an inverse problem for the heat equation with an involution, subject to Neumann boundary conditions. Theorems on the existence and uniqueness of the solution to this problem are presented. It is known that inverse problems in mathematical physics include problems of determining coefficients or the right-hand side of a given differential equation.

Introduction

Problem statement

Shall now proceed with the problem statement.

$\Omega = \{-\pi < x < \pi,\ 0 < t < T\}, \quad |\varepsilon| < 1$ located in a rectangular region

$$u_t(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(-x,t) = f(x), \quad (x,t)\epsilon\Omega \tag{1}$$

consider the linear heat conduction equation.

Inverse problem with Neumann boundary conditions

The solvability problems of the following inverse problem are studied Ω area (1) equation and

$$u(x,0)=\varphi(x),\ u(x,o) = \varphi(x),\ u(x,T) = \phi(x),\ x \in [-\pi,\pi] \tag{2}$$

Condition and

$$u(-\pi,t) = 0, \quad u(\pi,t) = 0, \quad t\epsilon[0,T] \tag{3}$$

Determine the functions $u(x,t)$ and $f(x)$ that fulfill the homogeneous Neumann boundary condition.

Here, $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

In the systematic solution of the inverse problem, we denote the functions $u(x,t)$ and $f(x)$ belonging to the class as $u(x,t)\in C^{2,1}_{x,t}(\Omega),$

Research methodology

Solution Method

We seek a solution to the inverse problem in the form of a series expansion using a system of orthogonal basis functions on $L_2(-\pi,\pi)$. To find the system of functions, we solve the corresponding homogeneous equation with boundary conditions given by equation (1) using the method of separation of variables.

Spectral Problem

Using the method of separation of variables, the inverse problem is reduced to the following spectral problem:

$$X''(x) - \varepsilon X''(x) + \lambda X(x) = 0,\ X'(-\pi) = X'(\pi) = 0 \tag{4}$$

(4) the eigenvalue problems are self-adjoint and therefore have real eigenvalues. The corresponding eigenfunctions form an orthogonal basis on $L_2(-\pi,\pi)$. The eigenvalues [2]

$$\lambda_{1k} = (1-\varepsilon)k^2,\ \lambda_{2k} = (1+\varepsilon)(k+\tfrac{1}{2})^2 \tag{5}$$

will be the eigenvalues, and the corresponding eigenfunctions are defined as follows:

$$X_0 = 1,\ X_{1k} = \cos kx, k\epsilon N X_{2k} = \sin\left(k+\tfrac{1}{2}\right)x, k\epsilon N \tag{6}$$

(6) The function is complete and orthogonal for $L_2(-\pi,\pi)$.

Existence of a solution:

Solutions of the inverse problem $u(x,t)$ and $f(x)$ can be represented as series expansions using the corresponding set of eigenfunctions. Using the orthogonal system (6), functions $u(x,t)$ and $f(x)$ can be written as follows

$$u(x,t) = \sum_{k=0}^{\infty} u_{k1}(t)\cos kx + + \sum_{k=1}^{\infty} u_{k2}(t)\sin\left(k+\tfrac{1}{2}\right)x \tag{7}$$

$$f(x) = \sum_{k=0}^{\infty} f_{1k}\cos kx + \sum_{k=1}^{\infty} f_{2k}\sin\left(k+\tfrac{1}{2}\right)x \tag{8}$$

Here, $u_{1k}(t), u_{2k}(t), f_{1k}, f_{2k}$ are unknown coefficients.

Substituting functions (7) and (8) into equation (1), we obtain the following equations $u_{1k}(t)$, $u_{2k}(t)$ and constants f_{1k}, f_{2k}.

$$u'_{1k}(t) + (1-\varepsilon)k^2 \cdot u_{1k}(t) = f_{1k} \tag{9}$$

$$u'_{2k}(t) + (1+\varepsilon)\left(k+\tfrac{1}{2}\right)^2 \cdot u_{2k}(t) = f_{2k} \tag{10}$$

By solving this system of equations and applying condition (2), we can determine the values of the unknown constants $C_{1k}, C_{2k}, f_{1k}, f_{2k}$,

$$u_{1k}(t) = \frac{f_{1k}}{(1-\varepsilon)k^2} + c_{1k}e^{-(1-\varepsilon)k^2 t}$$

$$u_{2k}(t) = \frac{f_{2k}}{(1+\varepsilon)\left(k+\frac{1}{2}\right)^2} + c_{2k}e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 t}.$$

Let's say that the coefficients $\varphi_{ik}, \psi_{ik}, i=1,2$ are respectively $\varphi(x)$ and $\psi(x)$, is expanded into a series, namely

$$\varphi_{1k} = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x)\cos kx dx$$

$$\varphi_{2k} = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x) \sin\left(k + \frac{1}{2}\right) x dx$$

$$\psi_{1k} = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x) \cos k x dx$$

$$\psi_{2k} = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x) \sin\left(k + \frac{1}{2}\right) x dx$$

Then condition (2) leads to the following expressions:

$$\frac{f_{1k}}{(1-\varepsilon)k^2} + c_{1k} = \varphi_{1k}$$

$$\frac{f_{1k}}{(1-\varepsilon)k^2} + c_{1k} e^{-(1-\varepsilon)k^2 T} = \psi_{1k}$$

$$\frac{f_{2k}}{(1+\varepsilon)\left(k + \frac{1}{2}\right)^2} + c_{2k} = \varphi_{2k}$$

$$\frac{f_{2k}}{(1+\varepsilon)\left(k + \frac{1}{2}\right)^2} + c_{2k} e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 T} = \psi_{2k}$$

By solving this system of algebraic equations, we obtain the following expressions:

$$c_{1k} = \frac{\varphi_{1k} - \psi_{1k}}{1 - e^{-(1-\varepsilon)k^2 T}}$$

$$f_{1k} = (1-\varepsilon)k^2(\varphi_{1k} - c_{1k})$$

$$c_{2k} = \frac{\varphi_{2k} - \psi_{2k}}{1 - e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 T}}$$

$$f_{2k} = (1+\varepsilon)\left(k + \frac{1}{2}\right)^2 (\varphi_{2k} - c_{2k})$$

$$f_{1k} = (1-\varepsilon)k^2\left(\varphi_{1k} - \frac{\varphi_{1k} - \psi_{1k}}{1 - e^{-(1-\varepsilon)k^2 T}}\right)$$

$$f_{2k} = (1+\varepsilon)\left(k + \frac{1}{2}\right)^2 \left(\varphi_{2k} - \frac{\varphi_{2k} - \psi_{2k}}{1 - e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 T}}\right)$$

From this, for the coefficients $u_{1k}(t)$, $u_{2k}(t)$, we obtain the following equalities:

$$u_{1k}(t) = \varphi_{1k} - \frac{\varphi_{1k} - \psi_{1k}}{1 - e^{-(1-\varepsilon)k^2 T}} + \frac{\varphi_{1k} - \psi_{1k}}{1 - e^{-(1-\varepsilon)k^2 T}} \cdot e^{-(1-\varepsilon)k^2 t} =$$

$$= \varphi_{1k} - \frac{\varphi_{1k} - \psi_{1k}}{1 - e^{-(1-\varepsilon)k^2 T}}\left(1 - e^{-(1-\varepsilon)k^2 t}\right)$$

$$u_{2k}(t)=\varphi_{2k}-\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}+\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}\cdot e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2t}=$$

$$=\varphi_{2k}-\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2t}\right)$$

Now, substituting functions $u_{1k}(t), u_{2k}(t), f_{1k}, f_{2k}$ into (7) and (8), we obtain a formal solution to the inverse problem.

$$u(x,t)=\sum_{k=0}^{\infty}\left[\varphi_{1k}-\frac{\varphi_{1k}-\psi_{1k}}{1-e^{-(1-\varepsilon)k^2T}}\left(1-e^{-(1-\varepsilon)k^2t}\right)\right]\cos kx+$$

$$+\sum_{k=1}^{\infty}\left[\varphi_{2k}-\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2t}\right)\right]\sin\left(k+\frac{1}{2}\right)x=$$

$$=\sum_{k=0}^{\infty}\varphi_{1k}\cos kx-\sum_{k=0}^{\infty}\frac{\varphi_{1k}-\psi_{1k}}{1-e^{-(1-\varepsilon)k^2T}}\left(1-e^{-(1-\varepsilon)k^2t}\right)\cos kx+$$

$$+\sum_{k=1}^{\infty}\varphi_{2k}\sin\left(k+\frac{1}{2}\right)x-$$

$$-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2t}\right)\sin\left(k+\frac{1}{2}\right)x=$$

$$=\varphi(x)-\sum_{k=0}^{\infty}\frac{\varphi_{1k}-\psi_{1k}}{1-e^{-(1-\varepsilon)k^2T}}\left(1-e^{-(1-\varepsilon)k^2t}\right)\cos kx-$$

$$-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2t}\right)\sin\left(k+\frac{1}{2}\right)x$$

and

$$f(x)=\sum_{k=0}^{\infty}\left[(1-\varepsilon)k^2(\varphi_{1k}-\frac{\varphi_{1k}-\psi_{1k}}{1-e^{-(1-\varepsilon)k^2T}})\right]\cos kx+$$

$$+\sum_{k=1}^{\infty}\left[(1+\varepsilon)\left(k+\frac{1}{2}\right)^2(\varphi_{2k}-\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}})\right]\sin\left(k+\frac{1}{2}\right)x$$

From the obtained series, when $k\to\infty$, the expression's

$$\frac{e^{-(1-\varepsilon)k^2t}-1}{1-e^{-(1-\varepsilon)k^2T}}\sim Ce^{-(1-\varepsilon)k^2(t-T)}\to\infty \text{ if } k\to\infty, t-T<0,$$

$$\frac{(1-\varepsilon)k^2}{1-e^{-(1-\varepsilon)k^2T}}\to\infty \quad \text{if} \quad k\to\infty, T>0$$

both the function $u(x,t)$ and the function $f(x)$ have first terms that form divergent series. Therefore, it is necessary to require that the equality $\varphi_{1k}-\psi_{1k}=0, k=0,1,\ldots$ holds. To ensure that these equalities hold, it is sufficient to require that the functions $\varphi(x)$ and $\psi(x)$ are odd. If this is the case, then...

$$\varphi_{1k}=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x)\cos k\,xdx=\frac{1}{\pi}\int_{-\pi}^{0}\varphi(x)\cos k\,xdx+$$

$$+\frac{1}{\pi}\int_{0}^{\pi}\varphi(x)\cos k\,xdx=\frac{1}{\pi}\int_{0}^{\pi}\varphi(-x)\cos k\,xdx+$$

$$+\frac{1}{\pi}\int_{0}^{\pi}\varphi(x)\cos k\,xdx=0,\qquad k\geq 0$$

Indeed, if that is the case, then

$$\varphi_{1k}=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi(x)\cos k\,xdx=\frac{1}{\pi}\int_{-\pi}^{0}\varphi(x)\cos k\,xdx+\frac{1}{\pi}\int_{0}^{\pi}\varphi(x)\cos k\,xdx$$

Further, after substituting $x=-t$, for the first integral

$$\frac{1}{\pi}\int_{-\pi}^{0}\varphi(x)\cos k\,xdx=\frac{1}{\pi}\int_{-\pi}^{0}\varphi(-t)\cos k\,(-t)d(-t)=$$

$$=\tfrac{1}{\pi}\textstyle\int_{0}^{\pi}\varphi(-t)\cos k\,tdt=-\tfrac{1}{\pi}\textstyle\int_{0}^{\pi}\varphi(x)\cos k\,xdx.$$

There is $\varphi_{1k}=0$.

Accordingly, $\psi_{1k}=0, k=0,1,\ldots$ is shown.

Then, the formal solution to the problem is as follows

$$u(x,t)=\varphi(x)-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}t}\right)\sin\left(k+\frac{1}{2}\right)x$$

and

$$f(x)=\sum_{k=1}^{\infty}\left[(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}(\varphi_{2k}-\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}T}})\right]\sin\left(k+\frac{1}{2}\right)x$$

Convergence of a series

To establish the correctness of the obtained formal solutions, we must demonstrate the uniform convergence on the domain Ω of the series involving the functions $u(x,t)$ and $f(x),$, as well as the derivatives of $u_{xx}(x,t)$ and $u_t(x,t)$. To this end, we require the following conditions:

$$\varphi^{(i)}(-\pi)=\varphi^{(i)}(\pi)=0,\quad i=0,2,$$

$$\psi^{(i)}(-\pi)=\psi^{(i)}(\pi)=0, \quad i=0,2.$$

Consequently, c_{1k}, C_{2k} can be expressed as follows:

$$c_{1k}=\frac{\varphi_{2k}^{(3)}-\psi_{2k}^{(3)}}{1-e^{-(1-\varepsilon)k^2T}k^3}$$

$$c_{2k}=-\frac{\varphi_{1k}^{(3)}-\psi_{1k}^{(3)}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}\left(k+\frac{1}{2}\right)^3}$$

here,

$$\varphi_{1k}^{(3)}=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi'''(x)\cos\left(k+\frac{1}{2}\right)xdx$$

$$\varphi_{2k}^{(3)}=\frac{1}{\pi}\int_{-\pi}^{\pi}\varphi'''(x)\sin k\,xdx$$

$$\psi_{1k}^{(3)}=\frac{1}{\pi}\int_{-\pi}^{\pi}\psi'''(x)\cos\left(k+\frac{1}{2}\right)xdx$$

$$\psi_{2k}^{(3)}=\frac{1}{\pi}\int_{-\pi}^{\pi}\psi'''(x)\sin k\,xdx$$

From this, it follows that the functions $u(x,t)$ and $f(x)$ can be expressed as follows:

$$u(x,t)=\varphi(x)+$$

$$+\sum_{k=1}^{\infty}\frac{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2t}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}\left(\frac{\varphi_{1k}^{(3)}-\psi_{1k}^{(3)}}{\left(k+\frac{1}{2}\right)^3}\right)\sin\left(k+\frac{1}{2}\right)x$$
$$-\sum_{k=0}^{\infty}\frac{1-e^{-(1-\varepsilon)k^2t}}{1-e^{-(1-\varepsilon)k^2T}}\left(\frac{\varphi_{2k}^{(3)}-\psi_{2k}^{(3)}}{k^3}\right)\cos k\,x$$

and

$$f(x)=-\varphi''(x)+\varepsilon\varphi''(-x)+\sum_{k=1}^{\infty}\frac{1+\varepsilon}{\left(k+\frac{1}{2}\right)}\left(\frac{\varphi_{1k}^{(3)}-\psi_{1k}^{(3)}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2T}}\right)\sin\left(k+\frac{1}{2}\right)x$$
$$-\sum_{k=0}^{\infty}\frac{1-\varepsilon}{k}\left(\frac{\varphi_{2k}^{(3)}-\psi_{2k}^{(3)}}{1-e^{-(1-\varepsilon)k^2T}}\right)\cos k\,x$$

To investigate the convergence of these series, we employ the following estimates for $u(x,t)$ and $f(x)$

$$|u(x,t)|\le|\varphi(x)|+$$

$$+c\sum_{k=1}^{\infty}\frac{\left|\varphi_{1k}^{(3)}\right|+\left|\psi_{1k}^{(3)}\right|}{\left(k+\frac{1}{2}\right)^{3}}+c\sum_{k=0}^{\infty}\frac{\left|\varphi_{2k}^{(3)}\right|+\left|\psi_{2k}^{(3)}\right|}{k^{3}}$$

and

$$|f(x)|\le|\varphi''(x)|+|\varphi''(-x)|+c\sum_{k=1}^{\infty}\left(\left|\varphi_{1k}^{(3)}\right|^{2}+\left|\psi_{1k}^{(3)}\right|^{2}+\frac{2}{\left(k+\frac{1}{2}\right)^{2}}\right)$$
$$+c\sum_{k=0}^{\infty}\left(\left|\varphi_{2k}^{(3)}\right|^{2}+\left|\psi_{2k}^{(3)}\right|^{2}+\frac{2}{k^{2}}\right)$$

Here, C is some positive constant.

Here, we have used the inequality $2ab\le a^2+b^2$ for estimation f(x). For the function $u(x,t)$ in the estimate, if the terms of the series $\varphi_{ik}^{(3)},\psi_{ik}^{(3)},i=1,2$ are bounded, then the convergence of the series is guaranteed. This condition is satisfied if we consider $\varphi'''(x)$ and $\psi'''(x)\in L_2(-\pi,\pi)$. Additionally, for trigonometric series, the following series converge according to Bessel's inequality:

$$\sum_{k=1}^{\infty}\left|\varphi_{ik}^{(3)}\right|^{2}\le C\,\|\varphi'''(x)\|_{L_{2(-\pi,\pi)}}^{2},\qquad i=1,2,$$
$$\sum_{k=1}^{\infty}\left|\psi_{ik}^{(3)}\right|^{2}\le C\,\|\psi'''(x)\|_{L_{2(-\pi,\pi)}}^{2},\qquad i=1,2.$$

Consequently, according to the Weierstrass M-test, the series $u(x,t)$ and $f(x)$ converge absolutely and uniformly on the domain Ω . Similarly, the series $u(x,t)$ obtained by term-by-term differentiation of the series $u_{xx}(x,t)$ and $u_t(x,t)$ its derivatives can be shown to converge in a similar manner.

Results of the study

Main Results

Let us formulate the main result related to problems (1)-(3).

Theorem 3.1. Let $\varphi(x),\psi(x)$ be odd functions, $\varphi(x),\psi(x)\in C^2[-\pi,\pi]$, $\varphi'''(x)$, $\varphi'''(x),\psi'''(x)\in L_2(-\pi,\pi)$ and $\varphi^{(i)}(\pm\pi)=\psi^{(i)}(\pm\pi)=0,\ i=0,1,2.$

Therefore, the fact that the inverse problem (3.1)-(3.3) has a unique solution can be expressed as follows:

$$u(x,t)=\varphi(x)-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}t}\right)\sin\left(k+\frac{1}{2}\right)x$$

$$f(x)=\sum_{k=1}^{\infty}\left[(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}(\varphi_{2k}-\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^{2}T}})\right]\sin\left(k+\frac{1}{2}\right)x$$

here

$$\varphi_{2k} = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x)\sin\left(k+\frac{1}{2}\right)xdx$$
$$\psi_{2k} = \frac{1}{\pi}\int_{-\pi}^{\pi} \psi(x)\sin\left(k+\frac{1}{2}\right)xdx$$

To prove the theorem, it is necessary to verify that the conditions of the problem are satisfied. Let's take the next one

$$u(x,0) = \varphi(x) -$$
$$-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2\cdot 0}\right)\sin\left(k+\frac{1}{2}\right)x = \varphi(x)$$

$$u(x,T) = \varphi(x) -$$
$$-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2\cdot T}\right)\sin\left(k+\frac{1}{2}\right)x =$$
$$= \varphi(x) - \sum_{k=1}^{\infty}\varphi_{2k}\sin\left(k+\frac{1}{2}\right)x + \sum_{k=1}^{\infty}\psi_{2k}\sin\left(k+\frac{1}{2}\right)x = \varphi(x) -$$
$$-\varphi(x) + \sum_{k=1}^{\infty}\psi_{2k}\sin\left(k+\frac{1}{2}\right)x = \psi(x)$$

Therefore, the initial and final conditions of the redefinition are satisfied.

Conclusion

Moreover, according to the theorem's condition, $\varphi(\pi)=\varphi(-\pi)=0$, therefore,

$$u(\pi,t) = \varphi(x) -$$
$$-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2\cdot t}\right)\sin\left(k+\frac{1}{2}\right)x\Bigg|_{x=\pi} =$$
$$= \varphi(\pi) = 0$$

$$u(-\pi,t) = \varphi(x) -$$
$$-\sum_{k=1}^{\infty}\frac{\varphi_{2k}-\psi_{2k}}{1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2 T}}\left(1-e^{-(1+\varepsilon)\left(k+\frac{1}{2}\right)^2\cdot t}\right)\sin\left(k+\frac{1}{2}\right)x\Bigg|_{x=-\pi} =$$
$$= \varphi(-\pi) = 0$$

that is, the boundary condition is also satisfied.
The theorem has been proven.

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