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A.R. Yeshkeyev[1], O.I. Ulbrikht[1], A.K. Koshekova[1]*

[1]Karaganda Buketov University, Karaganda, Kazakhstan
*e-mail: koshekova1998@mail.ru

PROPERTIES OF CATEGORICITY FOR KAISER CLASS

Abstract

In this article we present concrete results on both the countable and uncountable categoricity of some fragments. Our research on the categoricity of these fragments is conducted within the extended framework of normal Jonsson theory. The fragments are derived through the application of the closure operator of a specified pregeometry, and the involved sets are regular. The resulting models in this context constitute the Kaiser class associated with a studied Jonsson theory. These models exhibit distinct and interesting structural properties, making them a subject of extensive study. Furthermore, we analyze how variations in the underlying pregeometry influence the classification of fragments. Such an approach makes it possible to identify broader and deeper connections between categoricity and model-theoretic stability, thereby significantly expanding and refining the understanding of their interrelation in the context of contemporary fundamental scientific research.

Keywords: Jonsson theory, semantic model, Jonsson set, almost Jonsson set, normality, regularity.

А.Р. Ешкеев[1], О.И. Ульбрихт[1], А.К. Кошекова[1]
[1]Академик Е.А. Бөкетов атындағы Қарағанды университеті, Қарағанды қ., Қазақстан

КАЙЗЕР КЛАСЫ ҮШІН ӘЛБЕТТІЛІК (КАТЕГОРЛЫЛЫҚ) ҚАСИЕТТЕРІ

Аңдатпа

Бұл мақалада біз белгілі бір фрагменттердің саналатын және саналмайтын әлбеттілігі бойынша нақты нәтижелерді ұсынамыз. Бұл фрагменттердің әлбеттілігін зерттеу нормальді йонсондық теориясының кеңейтілген шеңберінде жүргізілді. Фрагменттер алғашқы геометрияның тұйықтама операторы арқылы алынады, ал қолданылған жиындар тұрақты болып табылады. Бұл жағдайда алынған модельдер зерттелген йонсондық теориясына сәйкес Кайзер класын құрайды. Бұл модельдер ерекше қызықты құрылымдық қасиеттерді көрсетеді және кеңінен зерттеу нысанына айналады. Сонымен қатар, негізгі алғашқы геометриядағы өзгерістердің фрагменттерді жіктеуге қалай әсер ететінін талдаймыз. Мұндай тәсіл әлбеттілік пен модельдік-теориялық тұрақтылық арасындағы анағұрлым кең әрі терең байланыстарды айқындауға мүмкіндік береді, осылайша олардың өзара қатынасын қазіргі іргелі ғылыми зерттеулер аясында едәуір кеңейтіп, тереңдетіп, нақтылай түседі.

Түйін сөздер: йонсондық теория, семантикалық модель, йонсондық жиын, йонсондық дерлік жиын, қалыптылық, тұрақтылық.

А.Р. Ешкеев[1], О.И. Ульбрихт[1], А.К. Кошекова[1]
[1]Карагандинский университет имени академика Е.А. Букетова, г. Караганда, Казахстан

СВОЙСТВА КАТЕГОРИЧНОСТИ ДЛЯ КЛАССА КАЙЗЕРА

Аннотация

В данной статье мы представляем конкретные результаты исследований о счетной и несчетной категоричности некоторых фрагментов. Изучение категоричности этих фрагментов проводится в расширенных рамках нормальной йонсоновской теории. Фрагменты получаются с использованием оператора замыкания заданной предгеометрии, а рассматриваемые множества являются регулярными. Полученные модели в данном случае образуют класс Кайзера, связанный с изучаемой йонсоновской теорией. Эти модели демонстрируют особые интересные структурные свойства, что делает их предметом активного изучения. Кроме того, мы анализируем, как изменения в базовой предгеометрии влияют на классификацию фрагментов. Подобный подход позволяет выявить более широкие и глубокие связи между категоричностью и модельно-теоретической стабильностью, тем самым существенно расширяя и уточняя понимание их взаимосвязи в контексте современных фундаментальных научных исследований.

Ключевые слова: йонсоновская теория, семантическая модель, йонсоновская множество, почти йонсоновская множество, нормальность, регулярность.

Introduction

The main results of the paper are given in the form of new theorems exploring the relationships between key properties of a perfect normal Jonsson theory M^0, such as completeness, model completeness, ∀∃-axiomatizability, and the existence of a model companion. The focus is on how structures related to $cl(A) \in K_T$ and M^0 ensure the equivalence of these fundamental properties within the framework of Jonsson theory. The results establish a profound connection between the fundamental properties of a perfect normal Jonsson theory M^0. Specifically, it has been shown that the properties of completeness, model completeness, ∀∃-axiomatizability, and the existence of a model companion are all equivalent within the framework of Jonsson theory. Furthermore, the structures related to $cl(A) \in K_T$ and M^0 play a crucial role in ensuring this equivalence.

This article examines the model-theoretic characteristics of particular subsets within the semantic models of a given fixed normal Jonsson theory, with a primary focus on regular almost Jonsson sets. The foundational concepts of normality for Jonsson theories and almost Jonsson sets, first introduced in [1], serve as the basis for this investigation. Regularity is presented as a natural and essential requirement for definable sets to qualify as Jonsson sets. To enable effective analysis, the defining formula of a Jonsson set must satisfy established model-theoretic criteria. In this study, the property of regularity is axiomatically formulated, with the defining set characterized by a formula possessing a Morley rank. This approach integrates seamlessly with the broader framework of studying Jonsson theories in the context of Morley rank, as previously developed in [2].

Building on foundational works [1], [3], and [4], this article addresses the challenges posed by the general incompleteness of Jonsson theories. In the absence of a comprehensive analysis of the Lindenbaum-Tarski Boolean algebra of formulas and its associated Stone space of types, alternative methods are employed. Following the approaches of [3] and [4], this work examines the lattice of existential formulas and their corresponding existential types. Within this framework, significant results concerning the structure and properties of almost Jonsson sets are obtained.

The key contribution of this study lies in the application of a new double factorization technique to the class of cosemanticness under consideration. This new approach sheds new light on the analysis of Jonsson theories and their related subsets, building on existing results and presenting a fresh perspective.

Research Methodology

1. Essential details regarding the study of Jonsson theories.

To grasp the content and subtleties of our subsequent discussion, it is essential to revisit the key definitions and propositions linked to those presented in [1]. In this article, we work within a countable first-order language, and all the theories under consideration will likewise be countable.

Before introducing the concept of Jonsson theories, we first recall the definitions of two key properties that play a crucial role in analyzing this class of incomplete theories.

Definition 1.

[5]. A theory T has the joint embedding property (JEP), if, for any models A and B of T, there exists a model M of T and isomorphic embeddings $f: A \to M$, $g: B \to M$.

Definition 2.

[5]. A theory T has the amalgamation property (AP), if for any models A, B_1, B_2 of T and isomorphic embeddings $f_1: A \to B_1, f_2: A \to B_2$ there are $M \models T$ and isomorphic embeddings $g_1: B_1 \to M, g_2: B_2 \to M$, such that $g_1 \circ f_1 = g_2 \circ f_2$.

The primary focus of this article revolves around the concept of a Jonsson theory.

Definition 3.

[5]. A theory T is said to be a Jonsson theory if:

(i) T has at least one infinite model;

(ii) T is an inductive theory, meaning it can be axiomatized by $\forall\exists$-sentences;

(iii) T has JEP (joint embedding property), i.e. any two models of T can be combined into a larger model of T;

(iv) T has AP (amalgamation property), i.e. guarantees that for any three models of T, where two extend the third, there exists a larger model into which both extensions can be embedded consistently.

The study of Jonsson theory has been a subject of interest and recognition for many years. This framework, which captures the core methodologies and their applications in exploring various aspects of Jonsson theories, is detailed in the works [6]-[16].

The following definition introduces an important concept: an existentially closed model within the class of models of a Jonsson theory. When an existentially closed model is also algebraically simple, it gives rise to a particularly interesting class of Jonsson theories. This combination of properties contributes to the structural richness and definability of the theory, making it a key area of study in the model theory of Jonsson theories.

Definition 4.

[5]. Let M be a structure and $N \supseteq M$. A model M of theory T is existentially closed in N, if for any tuple $\bar{a} \in M$ and any quantifier-free formula $f(\bar{x}, \bar{y})$ of language M, the following holds: if $N \models (\exists\bar{y})f(\bar{a}, \bar{y})$, then $M \models (\exists\bar{y})f(\bar{a}, \bar{y})$.

This property reflects the closure of M under existential quantification within the extension N.

One of the syntactic invariants of a Jonsson theory is its center. This center serves as a key structural element in understanding the theory and its associated models, often reflecting important properties related to the theory's definability and consistency. Let us give some definitions regarding the Jonsson theories.

Definition 5.

[17]. A model C_T of the Jonsson theory T such that $|C_T| = 2^{\omega}$ is said to be a semantic model, if it is ω^{+}-homogeneous-universal.

Definition 6.

[18]. T^{*} denotes the center of a Jonsson theory T. It is defined as the elementary theory corresponding to its semantic model C_T, meaning that T^{*} is the complete theory that fully describes the model C_T. Formally, we can express this as: $T^{*} = Th(C_T)$, where $Th(C_T)$ is the theory of the model C_T, consisting of all the sentences that are true in C_T.

Definition 7.

[5]. Let T be an arbitrary theory of the language L. A theory T' is called a model companion of T, if it satisfies the following conditions:

(i) Mutual model-consistency: It means that any model of T is embedded in the model of T' and vice versa;

(ii) Model-completeness: T' is model-complete, meaning that for any two models B_1 and B_2 of T', if B_1 is a submodel of B_2, then B_1 is an elementary submodel of B_2. In simpler terms, any embedding of one model into another is elementary, preserving the truth of all formulas.

The concept of model completeness is intrinsically connected to the idea of mutual model consistency for the theory in question.

Definition 8.

[5]. Let T be any theory. A theory $T^{\#}$, referred to as a #-companion of T, is a theory of the same signature if:

(i) $(T^{\#})_{\forall} = T_{\forall}$: The universal consequences of $T^{\#}$ are identical to those of T;

(ii) if $T_{\forall} = T'_{\forall}$, then $T^{\#} = (T')^{\#}$: The #-companion is uniquely determined by the universal consequences of T;

(iii) $T_{\forall\exists} \subseteq T^{\#}$.

It is evident that a model companion is also a #-companion of T. This implies that the model companion satisfies the conditions required for a $T^{\#}$-companion, including:

(i) Identical Universal Consequences: The universal consequences of T and its model companion coincide, i.e., $T'_{\forall} = T_{\forall}$.

(ii) Inclusion of Existential Consequences: The model companion encompasses all the existential consequences of T, ensuring consistency with T while extending its structure.

This dual compatibility confirms that the model companion meets the broader requirements of a $T^{\#}$-companion.

Remark 1. When we state that $T'_{\forall} = T_{\forall}$, it means that every model of T can be embedded as a substructure within a model of T', and conversely, every model of T' can be embedded as a substructure within a model of T.

Moreover, T' is a model companion of T if and only if it is also a model companion of $T_{\forall}$.

Proposition 1. [5]. Let a theory T be an arbitrary theory. Then T has a model companion if and only if the class of existentially closed models of $T_{\forall}$ is an elementary class, and if a model companion of T exists, it is unique and is coincides with the theory of existentially closed models of $T_{\forall}$.

Additionally, Robinson's work demonstrates the concepts of finite forcing and the forcing #-companion. Robinson's result in [19] shows that if a theory in a countable language satisfies JEP, then this theory has a forcing #-companion that is complete.

The theorem below establishes that every Jonsson theory T has a forcing #-companion, and that this companion is a complete theory.

Theorem 1.

[5]. If T satisfies JEP, then the forcing #-companion $T^{\#}$ will be complete.

The theorem highlights that the simultaneous existence of all interpretations of the #-companion is intricately connected to the existence of a model companion.

This means that if a model companion exists for T, the different interpretations or versions of the #-companion is closely tied to the existence of a model companion. In simpler terms, the existence of a model companion guarantees that the #-companion can be well-defined and consistent across different interpretations.

In addition to the model companion, other types of companions are crucial in the analysis and understanding of Jonsson theories. These include the forcing companion, the existentially closed companion, and the Kaiser hull (the maximal ∀∃-theory that is mutually model-consistent with T). The Kaiser hull is particularly notable for its close relationship with the companion that characterizes the class of models that are existentially closed with respect to the given theory. It is also related to the companion that defines the class of generic models when investigating the forcing companion of the theory.

The natural interpretations of the #-companion $T^{\#}$ include the following theories:

(i) T^{*} is the center of the Jonsson theory T,

(ii) T^{f} refers to the forcing companion of the Jonsson theory T,

(iii) T^{M} denotes the model companion of the theory T.

(iv) $T^{e} = Th(E_T)$. T^{e} represents the existential closure companion, specifying the set of existentially closed models of T, with E_T denoting the class of all existentially closed models within the theory T.

If E_T stands for the class of T-existentially closed models of an inductive theory T, then E_T is guaranteed to be non-empty, as established in [18].

In the study of Jonsson theories, the class of perfect Jonsson theories holds a special significance. These theories are particularly important because they exhibit a high level of structural regularity and satisfy key conditions that make them central to understanding the broader class of Jonsson theories. Perfect Jonsson theories are often characterized by their well-behaved lattices of existential formulas, model-theoretic properties, and their ability to interact with other important concepts such as the #-companion and forcing companions.

Definition 9.

[18]. A Jonsson theory T is said to be perfect, if every model of T should be sufficiently saturated with respect to the model companion T^*, meaning it closely reflects the structure defined by T^*.

It turns out that the semantic model of a perfect Jonsson theory is closed under existential quantification, meaning that the model satisfies the property that any system of existential formulas that can be satisfied in any larger model of the theory can also be satisfied within the model itself.

Lemma 1.

[18]. Let C_T be the semantic model of a Jonsson theory T. Then C_T is existentially closed model of T.

The following theorem offers a criterion for determining whether a Jonsson theory is perfect:

Theorem 2. (Criterion of Perfectness).

[18]. Let T be any Jonsson theory. If T is perfect, then T^* is the model companion of theory T, and vice versa.

In the case of a complete Jonsson theory, the concept of a companion (whether it's a #-companion, model companion, or other) is identical to the center of T.

Corollary 1.

In the case of a perfect Jonsson theory T, the #-companion coincides with the center of T, meaning $T^{\#} = T^*$.

The concept of algebraic simplicity generalizes the notion of a simple model. It is defined by the fact that not every theory has an algebraically simple model. A general criterion for the existence of algebraically simple models has not yet been established. Additionally, the concept of a rigid model is closely related to algebraic simplicity, as both are concerned with specific structural properties of models in the context of a given theory.

Definition 10.

[20]. A model of T is called algebraically prime if, for any other model of T, there exists an isomorphic embedding of the algebraically prime model into the other model.

Algebraically prime models are highly embedded and play a key role in understanding the structure of models of a theory.

Definition 11.

[21]. A model M of the signature of given theory T (called a structure M in this context) is called a core model if it is isomorphic (structurally identical) to the unique substructure within every model of T.

Definition 12.

[18]. A theory T is said to be an existentially prime if both classes E_T and AP are non-empty and have a non-empty intersection.

The following concept, introduced by A. Robinson, is closely connected to all the theories discussed above.

Definition 13.

[22]. A theory T is called convex if, for any model $\mathfrak{A} \models T$ and any collection $\{\mathfrak{B}_i | i \in I\}$ of substructures of $\mathfrak{A}$, where each $\mathfrak{B}_i \models T$, the intersection $\bigcap_{i\in I} B_i$ is also a model of T, provided it is non-empty. A theory T is strongly convex if the intersection is always non-empty and still a model of T.

Let T be a theory that is strongly convex, Jonsson, and perfect, and that is complete for existential sentences in the language L. The combination of these properties ensures that T has a well-defined structure of models, with a unique core model and robust behavior in terms of existentially closed models and intersections of substructures. The completeness for existential sentences ensures that T captures all existential properties in its models, further strengthening the theory's foundational aspects.

Definition 14.

[17] A pregeometry (or matroid) (X, cl) is a set X with a closure operator $cl: \mathfrak{B}(X) \to \mathfrak{B}(X)$, where $\mathfrak{B}$ denotes the Boolean of a set such that for all $A \subseteq X$ and $a, b \in X$:

a) (Reflexivity) $A \subseteq cl(A)$.

b) (Finite character) $cl(A)$ is the union of all $cl(A')$, where A' runs through all finite subsets of A.

c) (Transitivity) $cl(cl(A)) = cl(A)$.

d) (Replacement lemma) $a \in cl(Ab) \backslash cl(A) \Rightarrow b \in cl(Aa)$.

A set A is called closed (or cl-closed) if $A = cl(A)$. Note that the closure $cl(A)$ of the set A is the smallest cl-closed set containing A. Therefore, the pregeometry is defined by a system of cl-closed subsets. The operator $cl(A) = A$ for all $A \subseteq X$ is a trivial example of pregeometry. Three standard examples from algebra: vector spaces with a linear closure operator, for a field K with a simple field F, with respect to an algebraic closure $cl(A) = F(A)^{alg} \cap K$, and for the field K of characteristic p, the p-closure is given by $cl(A) = K^p(A)$.

Later in the article, by the theory of T, we mean a strongly convex, perfect Jonsson theory, which is complete for existential sentences in the language L. The framework involves a pregeometry [17] defined by the closure operator cl on the set of all subsets of C_T.

In a strongly convex, Jonsson, and perfect theory T, the concept of a Jonsson set A is closely tied to the structure of existentially closed submodels of semantic models. The set A is definable within the theory T, and its closure under the closure operator cl results in an existentially closed submodel N, which captures the model-theoretic characteristics of the set A. This interplay between definability, closure, and existential closure is crucial for understanding the internal structure of models in the context of a Jonsson theory.

The notation $Th_{\forall\exists}(N)$ refers to the sets of all $\forall\exists$-sentences, that are true in the model N. These sentences are those that are universally quantified and involve existential quantifiers in their structure. The collection $Th_{\forall\exists}(N)$ represents the set of all sentences that describe the properties of the existentially closed model N in terms of the theory T, specifically focusing on the existential aspects of the model.

Definition 15.

[17] The Jonsson set is a subset A of the semantic model C of the Jonsson theory T, which has special properties that make it possible to study in detail the structure of the models of this theory.

Definition 16.

[4]. The fragment $Fr(X)$ of the Jonsson set X is a Jonsson theory obtained as a $\forall\exists$-sentences which are true in the model N which is a closure of this set, such that $Fr(X) = Th_{\forall\exists}(N)$.

The following definition provides a crucial extension of the concept of a Jonsson set.

Definition 17.

[1]. A set X is called almost Jonsson if it fulfills these conditions:

1) X is a definable subset of C_T, where C_T is the semantic model of Jonsson theory T;

2) $cl(X) = M \in Mod(T)$.

Furthermore, $Th_{\forall\exists}(M) = M^0 = Fr(X)$, and $M^0 \in JSp(C_T)$, where $JSp(C_T)$ denotes the Jonsson spectrum of C_T, i.e. $JSp(C_T) = \{T' | \, T'$ is a Jonsson theory and $C_T \models T'\}$.

This concept can be illustrated by an example of an arbitrary abelian group, which turns out to be the closure of some existential formula defining an almost Jonsson set in this abelian group.

The concept of normality for a Jonsson theory is defined for a class of theories where any fragment of their semantic models belongs to the Jonsson spectrum of C_T of the given Jonsson theory.

The following definition describes specific subsets of the semantic model of the given Jonsson theory. These subsets are important in understanding the structure and behavior of the models of the theory, particularly in relation to how they fit into the broader framework of the Jonsson spectrum and the properties of T.

Definition 18.

[1]. The Jonsson theory T is said to be normal if for any $X \subseteq C_T$, such that $cl(X) = M \in Mod(T), Fr(X) = M^0 \in JSp(C_T)$ and $C_{M^0} \preceq_{\exists_1} C_T$, where C_{M^0} is a semantic model of M^0.

An example of a normal theory is the universal theory of all unars. This theory is characterized by the fact that it has an empty list of axioms and is Jonsson.

These subsets are characterized by the fact that the formula that defines them has Morley rank [17]. Morley rank is a key concept in model theory that measures the complexity of definable sets. The following six properties are actually six axioms that define the Morley rankability properties, which govern how sets and their definability can be ranked in terms of complexity.

Let us introduce the concept of a family of regular subsets of the semantic model C_T, denoted by $Reg(C_T)$.

Definition 19.

A subset $X_i \subseteq C_T$ is called regular if X_i satisfies the following properties:

1) For each $i \in I$ the inclusion $X_i \in Reg(C_T)$ holds;
2) The set $Reg(C_T)$ is closed under finite Boolean combinations, i.e. from the inclusions $A, B \subseteq C_T^n$, $A, B \in Reg(C_T)$ it follows that $A \cup B \in Reg(C_T), A \cap B \in Reg(C_T)$ and $C_T^n / A \in Reg(C_T)$;
3) The set $Reg(C_T)$ is closed under the Cartesian product, i.e. from the inclusions $A, B \in Reg(C_T)$ it follows that $A \times B \in Reg(C_T)$;
4) The set $Reg(C_T)$ is closed under the projection, i.e. if $A \subseteq C_T^{n+m}$, $A \in Reg(C_T)$, $\pi_n(A)$ is the projection of the set A onto C_T^n, then $\pi_n(A) \in Reg(C_T)$;
5) The set $Reg(C_T)$ is closed under specialization, i.e. if $A \in Reg(C_T), A \subseteq C_T^{n+m}$ and $c \in C_T^n$, then $A(\underline{c}) = \{(\underline{c}, \underline{a}) \in A\} \in Reg(C_T)$;
6) The set $Reg(C_T)$ is closed under permutation of coordinates, i.e. if $A \in Reg(C_T), A \subseteq C_T^n$, and σ is a permutation of the set $\{1, \ldots, n\}$, then $\sigma(A) = \{(a_1, \ldots, a_n) \in A\} \in Reg(C_T)$.

In the context of a Jonsson theory, the regular subsets of the semantic model C_T are subsets that exhibit specific properties in relation to the closure operator and definability. Regular sets are those that satisfy closure under various set-theoretic operations, preserving their regularity in the context of definable subsets of C_T.

Lemma 2. Let us consider a subset of existentially closed models K from a class of models of normal Jonsson theory T and the set of all $\forall\exists$-sentences true in the models of K. Then, the theory $Th_{\forall\exists}(K)$ is a normal Jonsson theory.

This is a useful result in model theory as it demonstrates how properties of existential closure and $\forall\exists$-sentences are preserved in the context of Jonsson theories.

Proof. Let $K \subseteq E_T$, T be a normal Jonsson theory. Since T is a normal Jonsson theory, then T has JEP, then according to Theorem 3 $\forall A, B \in E_T \Rightarrow T^0(A) = T^0(B)$.

Theorem 3. [23] Let there exist two existentially closed models A and B of a theory T that satisfy JEP, then any $\forall\exists$-sentence true in one model must also be true in the other model.

This follows from the preservation of existential closure and the properties of the JEP, ensuring that existentially closed models of a Jonsson theory (and in general, models satisfying the joint embedding property) share consistent truth conditions for $\forall\exists$-sentences. And now let us define the concept of the Kaiser class of models for the arbitrary of the Jonsson theory. The Kaiser class K_T is defined as $K_T = \{A \in ModT | Th_{\forall\exists}(A)$ is a Jonsson theory$\}$.

Proposition 2. It is clear that $E_T \subseteq K_T$.

1. $K_T \neq \emptyset$ (if $T = T^* = Th(C_T)$ and T is perfect then $K_T = E_T$).
2. $\forall M \subseteq K_T, Th_{\forall\exists}(M)$ is a Jonsson theory.

The Jonsson sets and almost Jonsson sets play a key role in the generation of fragments in the context of a normal Jonsson theory. Specifically, when a Jonsson theory is normal, these sets generate fragments from the models of the Kaiser class of the theory. These generated fragments are cosemantic with the Jonsson theory under consideration.

2. Classification of fragments by categoricity.

Definition 20.

[20] φ is a Δ-formula with respect to the theory T if it can be expressed in terms of two existential formulas ψ_1 and ψ_2 that are both logically equivalent to φ in the models of T.

Existential formulas are often called $\exists$-formulas.

Therefore, Δ -formulas are invariant under embeddings between models of theory T. Along with the $\exists$-formulas, they constitute the main classes used to define relations in algebraically prime models.

Definition 21.[20].

(i) $(A, a_0, a_1, \ldots, a_{n-1}) \Rightarrow_\Gamma (B, b_0, b_1, \ldots, b_{n-1})$ means that the formula φ is true in A with parameters $\bar{a}$ if and only if it holds in B with the corresponding parameters $\bar{b}$.

(ii) $(A, \bar{a}) \equiv_\Gamma (B, \bar{b})$ holds if both of the following conditions are satisfied:

1. $(A, \bar{a}) \Rightarrow_\Gamma (B, \bar{b})$;

2. $(B, \bar{b}) \equiv_\Gamma (A, \bar{a})$.

As classes, we will consider either $\triangle$ or $\exists$.

Definition 22.

[20]. A formula $\varphi(x_1, x_2, \ldots, x_n)$ is said to be complete for Γ-formulas if the following conditions hold:

(i) φ must be consistent with the theory T.

(ii) for every formula $\psi(x_1, x_2, \ldots, x_n)$ in the class Γ that has the same free variables as φ, one of the following must holds: $T \vDash \forall \bar{x}(\varphi \to \psi)$, which means that T proves that φ implies ψ for all possible values of the variables $\bar{x}$, or T proves that φ implies the negation of ψ for all values of $\bar{x}$, $T \vDash \forall \bar{x}(\varphi \to \neg\psi)$.

Definition 23.

[20]. A model $\mathfrak{A}$ is called a Γ_1, Γ_2-atomic model of a theory T if the following conditions hold: $\mathfrak{A} \vDash T$, which implies that all sentences in T are satisfied in $\mathfrak{A}$; and for every finite n-tuple of elements from the universe of $\mathfrak{A}$, there exists a formula $\psi(x) \in \Gamma_1$ such that $\psi(x)$ is complete for Γ_2-formulas in the sense that, for any formula $\theta(\bar{x}) \in \Gamma_2$, either $T \cup \{\psi(\bar{x})\} \vDash \theta(\bar{x})$ or $T \cup \{\psi(\bar{x})\} \vDash \neg\theta(\bar{x})$ and $\psi(\bar{x})\}$ holds for $\bar{a}$ in $\mathfrak{A}$.

Let M^0 be a perfect normal strongly convex fragment complete for existential sentences. Then the following properties holds:

(i) M^0 is an existentially prime normal theory;

(ii) in particular, M^0 has a core model. By [10], this core model is $(\Delta, \exists)$-atomic.

To establish the groundwork for Theorem 9, we introduce the following key definitions and results:

Theorem 4. (Saracino) [24]. Let T be a complete theory in a countable language. Then the theory T is ω-categorical if and only if the theory T has a model companion T^*, and T^* is ω-categorical.

Definition 24.

[20]. Σ-nice and Σ^*-nice algebraically prime models:

(i) A model $\mathfrak{A}$ is called a Σ-nice algebraically prime model of the theory T if $\mathfrak{A}$ is a countable model of T and for every model $\mathfrak{B}$ of T and for any tuple $\bar{a}$ in $\mathfrak{A}$ and $\bar{b}$ in $\mathfrak{B}$, if $(A, a_0, a_1, \ldots, a_{n-1}) \to_\exists (B, b_0, b_1, \ldots, b_{n-1})$ (i.e., the $\exists$-quantifier embedding relation holds between the tuples in the two models), then for every $a_n \in A$, there exists a corresponding $b_n \in B$ such that: $(A, a_0, a_1, \ldots, a_n) \to (B, b_0, b_1, \ldots, b_n)$.

(ii) A model $\mathfrak{A}$ is called a Σ^*-nice algebraically prime model of T if $\mathfrak{A}$ is countable model and for every model $\mathfrak{B}$ of T, every $n \in \omega$, and for all $a_0, a_1, \ldots, a_{n-1} \in A, b_0, b_1, \ldots, b_{n-1} \in B$, if $(A, a_0, a_1, \ldots, a_{n-1}) \equiv_\exists (B, b_0, b_1, \ldots, b_{n-1})$, then for every $a_n \in A$ there exists some $b_n \in B$ such that $(A, a_0, a_1, \ldots, a_n) \equiv_\exists (B, b_0, b_1, \ldots, b_n)$.

Remark 2. As established in [20], an $\exists$-complete, perfect, strongly convex fragment both possesses a core model and serves as a core model itself. Moreover, the $(\Delta, \exists)$-atomicity of this fragment implies (Σ, Σ)-atomicity, as shown in [20].

This is a stronger atomicity condition that indicates the fragment is highly structured in terms of both existential and universal quantification, making it a foundational building block for the model-theoretic analysis of Jonsson theories.

Theorem 5. [20]. If two models of T are countable and (Σ, Σ)-atomic, then these two models are isomorphic.

To proceed with the proof of Theorem 13, we need to introduce several important concepts and results that will help structure the argument.

Definition 25.

[20]. A theory T is said to satisfy R_1 if for any existential formula $\varphi(\bar{x})$ consistent with T there exists a formula $\psi(\bar{x}) \in \Delta$, also consistent with T, such that $T \vDash \psi \rightarrow \varphi$.

Definition 26.

[20]. A countable model $\mathfrak{M}$ of a theory T is called countably algebraically universal if, for every countable model $\mathfrak{N}$ of T, there exists an isomorphism $\mathfrak{N} \rightarrow \mathfrak{M}$.

Theorem 6. [20]. Let T be a $\forall\exists$-theory complete for existential sentences, and assume that T satisfies R_1. Then, there is an equivalence between the existence of models with certain atomic properties – such as an algebraically prime model, $(\exists, \Delta)$ -atomic model and $(\Delta, \exists)$ -atomic model, Δ-nice algebraically prime model and the existence of a unique algebraically prime model.

The equivalence of the conditions discussed above highlights that a complete $\forall\exists$-theory that satisfies R_1 exhibits strong and consistent structural properties, particularly in terms of the atomicity and uniqueness of its models.

M. Morley's criterion (see [25]) for the ω_1-categoricity of a complete theory is a well-known result in model theory. The criterion provides a necessary and sufficient condition for a complete theory to be categorically unique in all uncountable models, which is a significant aspect of the theory's structure and its models.

Definition 27.

[20]. A model $\mathfrak{M}$ is said to be a proper prime elementary extension of $\mathfrak{N}$ if $\mathfrak{M} \supsetneqq \mathfrak{N}$ and for any model $\mathfrak{K}$ such that $\mathfrak{K} \supsetneqq \mathfrak{N}$, it follows that $\mathfrak{M} \prec \mathfrak{K}$.

Next theorem provides a criterion for ω_1-categoricity based on the existence of proper prime elementary extensions, linking model-theoretic structure to categoricity.

Theorem 7. (Morley) [25]. If T is ω_1-categorical, then every countable model of T can be extended to a proper prime elementary extension. Conversely, if every countable model of T has a proper prime elementary extension, then T must be ω_1-categorical.

In the framework of Jonsson theories, we offer an analogous version of Definition 27. Both deal with prime extensions, but Definition 27 applies to elementary extensions in general model theory, while Definition 28 specifically addresses extensions in Jonsson theories.

Definition 28.

[18]. Let $\mathfrak{M}$ and $\mathfrak{N}$ be models from E_T, where $\mathfrak{M} \subsetneq \mathfrak{N}$. We say that $\mathfrak{N}$ is an algebraically prime model extension of $\mathfrak{M}$ within E_T if every isomorphic embedding of $\mathfrak{M}$ into any model $\mathfrak{K} \in E_T$ can be extended to an isomorphism $\mathfrak{N} \longrightarrow \mathfrak{K}$.

To refine Theorem 7 within the context of studying fragments of Jonsson theories, we need to focus on the specific structure of Robinson theories, which are universally axiomatized Jonsson theories. These theories have a set of key properties, including the fact that they are ω_1-categorical and ω-stable, which helps us refine the notion of Morley rank.

Regular subsets of the semantic model C_T play a significant role in understanding the properties of the theory T. These subsets often represent well-behaved or structured portions of the model. The Morley rank provides a measure of complexity or definable structure within a model. It assigns a rank to definable sets, giving insight into their size and behavior in the context of a complete theory.

In Jonsson theories**,** fragments (subsets of the theory with specific logical properties) often exhibit interesting relationships with the model-theoretic notions like stability and rank.

The Morley rank of regular subsets in C_T helps analyze these fragments, providing a structured way to understand their properties within the semantic model.

Results of the study

In this section, we consider the issues related to the categoricity of the considered fragments of normal theories. Moreover, the fragments are generated by almost Jonsson regular subsets of the semantic model of the regarded Jonsson theory.

Let T be a normal Jonsson theory. And let A be a subset of the semantic model C_T, A is an almost Jonsson set, meaning $cl(A) = M \in K_T$, where K_T - the Kaiser class of the normal fixed Jonsson theory. $Th_{\forall\exists}(M) = Fr(A) = M^0$.

The following points describe the properties of these fragments.

Lemma 3. The fragment M^0 is a Jonsson theory.

This result is proven in the reference [18].

Let ${M^0}^{\#}$ be the #-companion of the fragment M^0.

Theorem 8. If $cl(A) \in K_T$, then the following conditions are equivalent: M^0 is perfect if and only if ${M^0}^{\#}$ is axiomatized by ∀∃-sentences.

This result is extracted in [18].

Theorem 9. Let $cl(A) \in K_T$. Then the following conditions are equivalent:

(i) M^0 is perfect,

(ii) M^0 has a model companion**.**

The proof of this equivalence follows from the criterion of perfectness.

The subsequent statement is easily verified.

Lemma 4. Let $cl(A_1) = M_1$, $cl(A_2) = M_2$, and both M_1, M_2 are elements of the class K_T. Suppose M_1, M_2 are Kaiser hulls of the normal Jonsson theory. Then M_1^0 and M_2^0 are mutually model consistent is equivalent to the equality of their #-companions.

Proof. If M_1^0 and M_2^0 are mutually model consistent, then $(M_1^0)_\forall = (M_2^0)_\forall$, and by the definition of the #-companion, the #-companions will be equal. Conversely, if the #-companions of A_1 and A_2 coincide, then $(M_1^0)_\forall = (M_2^0)_\forall$. By part (i) of the definition #-companion, $M_1^0 = (M_1^{\#})_\forall$ and $M_2^0 = (M_2^{\#})_\forall$. Consequently $(M_1^0)_\forall = (M_2^0)_\forall$. Thus, M_1^0 and M_2^0 are mutually model consistent.

It is well-known that the concepts of model completeness and the completeness of a theory do not generally coincide. However, Lindstrom's theorem [5] establishes a connection between these two concepts. The following theorem is related to Lindstrom's theorem on model completeness.

We are considering Jonsson fragments, specifically M^0, within a framework where $cl(A) \in K_T$. Here, $cl(A)$ represents the closure of a set A, and A is a subset of C_T of a Jonsson theory T. The theory T is existentially prime**,** strongly convex, and normal. These properties ensure that T has certain robust closure properties, particularly under intersections and embeddings of models.

Theorem 10. Let the closure of the set A, denoted $cl(A)$ belongs to certain class K_T, and let M^0 be a perfect normal Jonsson theory. Then M^0 is complete if and only if it is model complete, and vice versa.

Proof. Firstly we note that, from the perfectness of an existentially prime strongly convex Jonsson theory the perfectness of a fragment follows.

$(i) \Rightarrow (ii)$ Assuming M^0 is complete, we need to show that it is model complete.

Let M^0 be a complete fragment of the Jonsson set A. Consider the central completion of M^0, denoted $(M^0)^*$. Since $M^0 \subseteq (M^0)^*$ and M^0 is complete, we have $M^0 = (M^0)^*$. The model $(M^0)^*$ is the center of the fragment and is part of the normal Jonsson set, which is existentially prime and strongly convex. By the criterion of perfectness, M^0 is a perfect fragment. Therefore, the center of M^0 coincides with its model companion, which is model complete**.**

Thus, M^0 is model complete.

$(ii) \Rightarrow (i)$: The proof uses JEP of fragment M^0 and the model completeness of $(M^0)^\#$.

Suppose the opposite, i.e. M^0 is not complete, then there is a sentence φ such that neither φ nor $\neg\varphi$ is deducible in M^0. This would lead to two inconsistent sets: $M^0 \cup \varphi$ and $M^0 \cup \neg\varphi$. By the Joint Embedding Property (JEP), there would be two models, A_1 and A_2, satisfying $M^0 \cup \varphi$ and $M^0 \cup \neg\varphi$, respectively. These models A_1 and A_2 can be embedded isomorphically into a common model B by elementary embeddings $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$. However, this leads to a contradiction because $B \vDash \varphi \wedge \neg\varphi$. Therefore, M^0 must be complete.

The theorem below essentially says that under the conditions described, the model $\mathfrak{A}$ will satisfy all these atomic and nice properties, showing the interaction between them in this context.

Theorem 11. Let M^0 be an $\exists$-complete, perfect, normal, strongly convex Jonsson fragment, where $cl(A)$ belongs to K_T (means that the closure of some set A under the closure operator cl is a model in the class K_T of T-models) of some normal Jonsson set A, and let $\mathfrak{A}$ be a countable model of the theory M^0. It follows that $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$, where:

(i) $\mathfrak{A}$ is (Σ, Σ) −atomic,

(ii) $\mathfrak{A}$ is Σ^*-nice,

(iii) $\mathfrak{A}$ is existentially closed and Σ-nice.

The proof follows from Remark 1 and [19].

Theorem 12. Consider M^0 as a $\forall\exists$-complete, perfect, normal, and strongly convex Jonsson fragment of a normal Jonsson set A, where $cl(A) \in K_T$. Then the following statements are equivalent: if the #-companion of a Jonsson fragment M^0 is ω-categorical, then M^0 is ω-categorical, and vice versa.

Proof. $(i) \Rightarrow (ii)$ Assume that $(M^0)^\#$ is ω-categorical. From Theorem 1 (part ii), $(M^0)^\#$ is complete. By Theorem 4, $(M^0)^\#$ has a unique ω-categorical #-companion, denoted $(M^0)^\#$. Since $(M^0)^\#$ is model-consistent with M^0, it follows that the models of $(M^0)^{\#'}$ must be also consistent with M^0. Moreover, the model-completeness of $(M^0)^\#$ ensures that every formula in the language of $(M^0)^{\#'}$is equivalent to an $\exists$-formula. Applying Robinson's theorem on the uniqueness of model companions, and the criterion of perfection for a normal Jonsson theory, it follows that $(M^0)^\# = (M^0)^{\#'}$. Since $(M^0)^{\#'}$ is ω-categorical, it has a unique countable model N, which is countably saturated. This model N belongs to the class of models of M^0, as $Mod(M^0)^\# \subseteq Mod(M^0)^{\#'}$. By the perfection criterion of a normal Jonsson theory, N is also Σ^*-*nice*-model. Furthermore, $Mod(M^0)^\# = K^\#$, where $K^\#$ contains a unique (up to an isomorphism) countable model N, which is (L, L) −atomic as defined in Definition 24, where L is the full language. Therefore, N is also a (Σ_1, Σ_1)-atomic model of $(M^0)^\#$, due to the model-completeness of $(M^0)^\#$ (since $(M^0)^\# = (M^0)^{\#'}$). By the $\exists$-completeness of M^0, N is a (Σ_1, Σ_1)-atomic model of $(M^0)^\#$. Finally, by Theorem 11, N is a Σ^*-*nice*-model. Let B an arbitrary countable model of M^0, so $B \in Mod(M^0)^\#$ and $cardB = \omega$. Since M^0 is $\exists$-complete, it follows that $N \equiv_\exists B$ (this serves as the induction base). By the definition of Σ^*-*niceness*, we can inductively obtain $(N, a)_{a\in N} \equiv_\exists (N_1, f(a))_{a\in N}$, where f is a mapping such that $f(a) = a$ for any $a \in N$. Hence, $(N, a)_{a\in N} \equiv_\exists (N_1, f(a))_{a\in N}$, which implies $N \preceq_\exists B$. Thus, B is a (Σ_1, Σ_1)-atomic model. By Theorem 5, $B \cong N$. Since B was arbitrary, the fragment M^0 is ω-categorical.

$(ii) \Rightarrow (i)$ Now assume that M^0 is ω-categorical. Suppose, for the sake of contradiction, that $(M^0)^\#$ is not ω-categorical. If $(M^0)^\#$ were not ω-categorical, there exist non-isomorphic countable models N and B of $(M^0)^\#$. However, since $M^0 \subseteq (M^0)^\#$, the models N and B would also belong to the class of models of M^0 (i.e., $N, B \in ModM^0$). This contradicts the assumption that M^0 is ω-categorical. Thus, $(M^0)^\#$ must also be ω-categorical.

Theorem 13. Let $cl(A)$ belongs to K_T, where A is a regular set. Let M^0 be Jonsson fragment, which is an existentially prime, perfect normal Jonsson theory that is complete for the existential sentences of a Jonsson universal theory satisfying R_1. Then the structure $(M^0)^\#$ is ω_1-categorical if and only if any countable model in K_{M^0} possesses an algebraically prime model extension in K_{M^0}, and vice versa.

Proof. $(i) \Rightarrow (ii)$ Let $(M^0)^\#$ be ω_1-categorical. By Morley's theorem on uncountable categoricity, $(M^0)^\#$ is perfect. Then, by the criterion of the perfectness of a Jonsson theory, $(M^0)^\#$ is a model complete theory, and we have $Mod(M^0)^\# = K_{M^0}$. Since $(M^0)^\#$ is model-complete, any isomorphic embedding between its models is elementary. Moreover, as $(M^0)^\#$ is a complete theory, applying Theorem 2 yields the required statement.

$(ii) \Rightarrow (i)$ By Lemma 3.11.2 of [18] applied to the semantic model $\mathfrak{C}$ of an existential prime perfect Jonsson theory M^0, we know that $\mathfrak{C}$ is ω-universal. Typically, its cardinality exceeds countability. Thus, we consider a countable elementary submodel $\mathfrak{D} \subseteq \mathfrak{C}$. Since $\mathfrak{C}$ is existentially closed (by Lemma 3.11.3. of [18]), the submodel $\mathfrak{D}$ is also existentially closed. Consequently, $\mathfrak{D}$ is countably algebraically universal. By the assumption that M^0 is existentially prime, M^0, possesses an algebraically prime model $\mathfrak{A}_0$. We now construct a chain of algebraically prime model extensions $\{\mathfrak{A}_\delta\}$:

(i) Define $\mathfrak{A}_{\delta+1}$ as an algebraically prime model extension of $\mathfrak{A}_\delta$;

(ii) $\mathfrak{A}_\lambda = \cup\{\mathfrak{A}_\delta | \delta < \lambda\}$.

Let $\mathfrak{A} = \cup\{\mathfrak{A}_0\delta | \delta < \omega_1\}$. Suppose $\mathfrak{B} \vDash M^0$ and $card\mathfrak{B} = \omega_1$. To prove $\mathfrak{B} \approx \mathfrak{A}$, decompose $\mathfrak{B}$ into a chain $\{\mathfrak{B}_\delta | \delta < \omega_1\}$ of countable models. This decomposition is possible due to the existential prime Jonsson theory M^0.

We now define a function $g\colon \omega_1 \to \omega_1$ and construct a sequence of isomorphisms $\{f_\delta : \mathfrak{A}_{g\delta} \to \mathfrak{B}_\delta | 0 < \delta < \omega_1\}$ by induction on δ:

(i) $g_0 = 0$ and $f_0 : \mathfrak{A}_0 \to \mathfrak{B}_0$.

(ii) $g\lambda = \cup\{g\delta | \delta < \lambda\}$ and $f_\lambda = \cup\{f_\delta | \delta < \lambda\}$.

(iii) $f_{\delta+1}$ is equal to the union of the chain $\{f_\delta^\gamma | \gamma \leq \rho$ where ρ is determined by induction on γ.

(iv) $f_{\delta+1}^0 = f_\delta$, $f_{\delta+1}^\lambda = \cup\{f_\delta | \delta < \lambda\}$.

(v) Suppose that $f_1^\lambda\colon \mathfrak{A}_{g\delta+\gamma} \to \mathfrak{B}_{\delta+1}$. If $f_\delta^{\gamma+1}$is a mapping onto, then $\rho = \gamma$. Otherwise, by virtue of the algebraic primeness of $\mathfrak{A}_{g\delta+\gamma+1}$ we can continue $f_\delta^{\gamma+1}$ to $f_{\delta+1}^{\gamma+1}\colon \mathfrak{A}_{g\delta+\gamma+1} \to \mathfrak{B}_{\delta+1}$.

(vi) $g(\delta+1) = g\delta + \rho$.

It follows that $f = \cup\{f_\delta | \delta < \omega_1\}$ is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. By Theorem 12, since $\mathfrak{B}$ was an arbitrary model of M^0, and A is the unique algebraically prime and existentially closed model (by assumption and construction), K_{M^0} has a unique model of uncountable cardinality. Therefore, the semantic model of an existentially prime Jonsson theory M^0 is saturated, which implies that M^0 is perfect. Consequently, $Mod(M^0)^\# = K_{M^0}$, and $(M^0)^\#$ is ω_1-categorical.

Discussion

This work highlights the significance of ω_1-categoricity as a central property that connects the algebraic, existential, and structural aspects of existentially prime Jonsson theories. This relationship provides a unified framework for systematically studying the logical properties of these theories, offering both theoretical and practical insights for future research in model theory and the exploration of Jonsson theories.

Conclusion

The results presented reveal a notable connection between the model-theoretic properties of Jonsson theories and their respective fragments, especially in terms of ω_1-categoricity, algebraic primeness, and existential completeness. This connection enhances our understanding of how these properties influence the structure and behavior of models in Jonsson theories, demonstrating that ω_1-categoricity guarantees the uniqueness of countable models, while algebraic primeness and existential completeness ensure strong closure properties for these models under extensions and embeddings. Together, these properties provide a comprehensive framework for studying the logical foundations of the theory. Specifically, we have shown the equivalence of ω_1-categoricity and algebraic primeness, perfectness and model-completeness, and the $\forall\exists$-complete, perfect, normal, and strongly convex Jonsson fragment of a normal Jonsson set A. This analysis not only deepens our understanding of the structure of Jonsson theories but also paves the way for future research on the connections between algebraic and existential properties of such models. Furthermore, the results highlight the importance of ω_1-categoricity as a central concept for linking various model-theoretic aspects, offering new insights into the behavior and classification of models within these theories. In sum, the work emphasizes that ω_1-categoricity serves as a unifying property for understanding the model-theoretic behavior of existentially prime Jonsson theories, linking their algebraic, existential, and structural characteristics within a coherent and elegant framework.

All the essential concepts and statements related to these notions, which were not defined or discussed in the text of this article, can be found in the following list of references.

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