МАТЕМАТИКА ЖӘНЕ ФИЗИКАЛЫҚ ПРОЦЕСТЕР МЕН МЕХАНИКАЛЫҚ ЖҮЙЕЛЕРДІ МАТЕМАТИКАЛЫҚ МОДЕЛЬДЕУ

МАТЕМАТИКА И МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ФИЗИЧЕСКИХ ПРОЦЕССОВ И МЕХАНИЧЕСКИХ СИСТЕМ

MATHEMATICS AND MATHEMATICAL MODELING OF PHYSICAL PROCESSES AND MECHANICAL SYSTEMS

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THE SOLUTION OF CAUCHY PROBLEM OF THERMOELASTIC ROD DYNAMICS BY GENERALIZED FUNCTIONS METHOD

Abstract

The Cauchy problem is considered about the dynamics of a thermoelastic rod by the action of different force and heat effects under arbitrary initial conditions. Using the Green's tensor of the thermoelasticity equations, an analytical solution to the problem was obtained, allowing one to determine the deformations, stresses, and temperature in any cross-section of the rod and at any time, if its initial state and the acting force and heat sources are known. Numerical calculations of temperature and displacements under the action of pulsed concentrated force and heat sources are given. Calculations of the solution of the Cauchy problem under the action of regular force and heat sources distributed along the rod are performed. The developed program allows one to study thermodynamic processes in rods with various physical and mechanical parameters under the action of distributed, concentrated, and pulsed heat and force sources, described by singular generalized functions.

Keywords: thermoelastic rod, Cauchy problem, method of generalized functions, Green's tensor, displacement, temperature, heat flow.

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КОШИ ЕСЕБІНІҢ ТЕРМОСЕРПІМДІК ӨЗЕКШЕНІҢ ДИНАМИКАСЫН ЖАЛПЫЛАНҒАН ФУНКЦИЯЛАР ӘДІСІМЕН ШЕШУ

Аңдатпа

Бұл жұмыста термосерпімдік өзекшенің Коши есебінің кез келген күштік және жылулық әсерлер мен бастапқы шарттар кезінде қарастырылады. Термоупругтік теңдеулердің Грин тензоры көмегімен аналитикалық шешім алынды, бұл шешім өзекшенің кез келген көлденең қимасы мен кез келген уақыт мезетінде деформацияларды, кернеулерді және температураны есептеуге мүмкіндік береді, егер бастапқы жағдайлар мен күштік және жылулық көздер белгілі болса.Арнайы назар импульстік, шоғырланған күштік және жылулық көздер әсеріндегі температураның таралуы мен ығысуының сандық есептеулеріне бөлінген. Сонымен қатар, өзекшенің бойымен күштер мен жылу көздерінің таралуы мен тұрақты функциялар түрінде бөлінген жағдайлар бойынша есептеулер жүргізілді. Арнайы назар импульстік, шоғырланған күштік және жылулық көздер әсеріндегі температураның таралуы мен ығысуының сандық есептеулеріне бөлінген. Сонымен қатар, өзекшенің бойымен күштер мен жылу көздерінің таралуы мен тұрақты функциялар түрінде бөлінген жағдайлар бойынша есептеулер жүргізілді.

Түйін сөздер: термосерпімді өзек, Коши есебі, жалпыланған функция әдісі, Грин тензоры, орын ауыстыру, температура, жылу ағыны.

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РЕШЕНИЕ ЗАДАЧИ КОШИ ДИНАМИКИ ТЕРМОУПРУГОГО СТЕРЖНЯ МЕТОДОМ ОБОБЩЕННЫХ ФУНКЦИЙ

Аннотация

Рассматривается задача Коши динамики термоупругого стержня при произвольных силовых и тепловых воздействиях и начальных условиях. С помощью тензора Грина уравнений термоупругости было получено аналитическое решение задачи, позволяющее определить деформации, напряжения и температуру в любом сечении стержня и в любой момент времени, если известны его начальное состояние и действующие силовые и тепловые источники. Приведены численные расчеты температуры, перемещений при действии импульсных сосредоточенных силовых и тепловых источников. Проведены расчеты решения задач Коши при действии распределенных вдоль стержня силовых и тепловых источников регулярного вида. Разработанная программа позволяет исследовать термодинамические процессы в стержнях с различными физико-механическими параметрами при действии тепловых и силовых источников как распределенных, так и сосредоточенных, и импульсных, описываемых сингулярными обобщенными функциями.

Ключевые слова: термоупругий стержень, задача Коши, метод обобщенных функций, тензор Грина, перемещение, температура, тепловой поток.

Main provisions

Using the method of V.S.Vladimirov, a solution to the Cauchy problem for the equations of uncoupled thermoelasticity in the space of generalized functions of slow growth is constructed. Determination of the thermal stress state of a thermoelastic rod under the action of local and distributed force and thermal effects. The equations of mechanics of a deformable solid, methods of mathematical physics and the theory of generalized functions were used for the study. Analytical formulas for calculating the temperature, displacements, stresses and deformations of a rod under the action of local and distributed external force and thermal effects are constructed. The result of the conducted research is numerical calculations of the temperature and displacements of the rod under the action of pulsed concentrated force and heat sources.

Introduction

The development of theory and methods for solving thermoelasticity problems is crucial for numerous branches of technology and applied sciences. These problems emerge in the creation of new designs for steam and gas turbines, jet and rocket engines of high-speed aircraft, nuclear reactors, and mining equipment. Elements of these structures operate under uneven and non-stationary heating conditions, causing changes in the mechanical properties of materials and the emergence of temperature gradients. This results in unequal thermal expansion of individual parts of the structures [1-8]. Some materials become brittle when subjected to stress from a sharp gradient in a non-stationary temperature field and cannot withstand thermal shock. Repeated thermal stresses can lead to the destruction of structural elements, making the study of temperature field effects on the stress-strain state of structures essential. Rod elements are widely used across various technological fields. Numerous examples of loaded structural members include supports for structures, buildings, and bridges. In real conditions, dynamic loads can act on rods, causing vibrations that significantly impact the reliability of these rod elements and, consequently, the overall structure's reliability. An unsteady temperature field induces a time-varying deformation field in rod structures, affecting their strength and reliability during operation. Determining the thermally stressed state of rod structures, considering their mechanical properties (particularly elasticity), is a significant scientific and technical challenge. Mathematical modeling methods enable the study of physical processes within structures and their elements, allowing for the determination of dynamic characteristics during the design stage. This modeling provides a basis for predicting product behavior under specific operating conditions. Here the generalized solution to the Cauchy problem for thermoelastic rod is constructed under the action of nonstationary force loads and various types of heat sources by use the model of uncoupled thermoelasticity and generalized functions method. This method allows translating the initial boundary value problem for the system of thermoelasticity equations with initial conditions into the solution of the system of differential equations in the space of generalized vector-functions with a singular right-hand side. It contains singular generalized functions of the form of simple and double layers, the densities of which are determined by the initial conditions. The convolution of the Green's tensor of these equations with the right-hand side gives a solution in the space of generalized functions, and their regular integral representation gives a solution to the posed boundary value problem. Here, the Green's tensor of thermoelasticity equations was used, previously constructed in [9]. Regular integral representations of generalized solutions are obtained for given initial temperature, displacements and velocity of the rod. Using the Mathcad-15 program, numerical calculations of the Green's tensor for a system with dimensionless thermoelastic parameters are carried out. Solutions to Cauchy problems under the action of force loads and heat sources distributed along the rod are given.

Research methodology

Determination of the thermal stress state of a thermoelastic rod under the action of local and distributed force and thermal effects. The equations of mechanics of a deformable solid, methods of mathematical physics and the theory of generalized functions were used for the study.

Statement of the Cauchy problem for the equations of uncoupled thermoelasticity

We consider a thermoelastic rod, the equations of state of which have the form [10,11]:

$$\begin{cases} \rho c^2 u,_{xx} - \rho u,_{tt} - \gamma\theta,_x + \rho F_1(x,t) = 0 \\ \theta,_{xx} - \kappa^{-1}\theta,_t + F_2(x,t) = 0 \end{cases} \tag{1}$$

Where $x \in R^1, t \geq 0$. Here ρ is the mass density, *c is* the speed of propagation of elastic waves in the rod, γ is the thermal conductivity coefficient, $\kappa = \dfrac{k}{\rho c}$ the thermal diffusivity coefficient, $u(x,t)$ are the longitudinal movements of the sections of the rod, $\theta(x,t)$ is a relative temperature , $F_1(x,t)$ is longitudinal component of external force per unit length; $F_2(x,t)$ is a quantity characterizing power heat source. Here and below, partial derivatives are designated: $u_1 = u,\ u_2 = \theta,\quad u,_x = \partial_x u_i \triangleq \dfrac{\partial u_i}{\partial x},\quad u_i,_t = \partial_t u_i \triangleq \dfrac{\partial u_i}{\partial t}\quad (i = 1,2).$

The thermoelastic stress in the rod is determined by the Duhamel - Neuman law:

$$\sigma = \rho c^2 u,_x - \gamma\theta \tag{2}$$

The initial conditions are known:

$$u(x,0)=u_0(x),\quad u_{,t}(x,0)=v_0(x),\ \theta(x,0)=\theta_0(x),$$
$$u_0(x)\in C^2(R^1),\theta_0(x)\in C^2(R^1),v_0(x)\in C^2(R^1), \tag{3}$$

where $C^n(R^1)$ is the space of functions differentiable up to the n-th order on R^1.

We consider the Cauchy problem, the solution of which allows to determine the state of the rod at any time if its initial state and the operating power and heat sources are known.

It is required to find solutions to equations (1) with initial conditions (3), which satisfy the radiation conditions:

$$u(x,t)\to 0,\ \theta(x,t)\to 0 \text{ at } |x|\to\infty,\quad \forall t, \tag{4}$$

under the action of arbitrary forces and heat sources : $F_j(x,t)=L_1(R^1)$, at $j=1,2$.

Statement of the Cauchy problem in the space of generalized functions

To solve the problem we use the method of generalized function which V.S. Vladimirov used by solving the Cauchy problem for wave equations [12,13]. For this the following regular generalized functions are introduced:

$$\hat{u}(x,t)=u(x,t)H(t),\quad \hat{\theta}(x,t)=\theta(x,t)H(t),\quad \hat{F}_j(x,t)=F_j(x,t)H(t),\quad j=1,2. \tag{5}$$

where $u(x,t),\theta(x,t)$ are the solution to the Cauchy problem (1)-(4); $H(t)$ is the Heaviside step-function: $H(t)$= 1 for t >0, $H(t)$= 0 for t <0. That is, we define by zero the solution of the Cauchy problem outside the domain of definition (for t < 0). And let us consider the action of the differential operator of system (1) on these functions in the space of generalized functions $D'(R^2)$ [12,13].

To do this, at first we find generalized derivatives of these functions :

$$\frac{\partial\hat{u}}{\partial t}=\frac{\partial u}{\partial t}H(t)+u_0(x)\delta(t),\ \frac{\partial^2\hat{u}}{\partial t^2}=\frac{\partial^2 u}{\partial t^2}H(t)+v_0(x)\delta(t)+u_0(x)\delta'(t),$$
$$\frac{\partial\hat{\theta}}{\partial t}=\frac{\partial\theta}{\partial t}H(t)+\theta_0(x)\delta(t),$$
$$\frac{\partial\hat{u}}{\partial x}=\frac{\partial u}{\partial x}H(t),\ \frac{\partial^2\hat{u}}{\partial x^2}=\frac{\partial^2 u}{\partial x^2}H(t),\ \frac{\partial\hat{\theta}}{\partial x}=\frac{\partial\theta}{\partial x}H(t),\ \frac{\partial^2\hat{\theta}}{\partial x^2}=\frac{\partial^2\theta}{\partial x^2}H(t).$$

Here $\delta(t)$ is a singular δ-function. Then system (1) will $D'(R^2)$ take the form:

$$c^2\hat{u}_{,xx}-\hat{u}_{,tt}-\tilde{\gamma}\hat{\theta}_{,x}+F_1(x,t)H(t)=-v_0(x)\delta(t)-u_0(x)\delta'(t),$$
$$\hat{\theta}_{,xx}-\kappa^{-1}\hat{\theta}_{,t}+F_2(x,t)H(t)=\kappa^{-1}\theta_0(x)\delta(t), \tag{6}$$

$\tilde{\gamma}=\dfrac{\gamma}{\rho}$. These equations include the initial conditions as singular mass forces and heat sources.

Next we denote

$$\hat{F}_1(x,t)=F_1(x,t)H(t),\quad \hat{F}_2(x,t)=F_2(x,t)H(t),$$
$$\hat{G}_1(x,t)=\hat{F}_1(x,t)+v_0(x)\delta(t)+u_0(x)\delta'(t), \tag{7}$$
$$\hat{G}_2(x,t)=\hat{F}_2(x,t)-\kappa^{-1}\theta_0(x)\delta(t).$$

The solution to this problem in the space of generalized functions has the form of tensor-functional convolution ($*$):

$$\hat{u}(x,t) = u(x,t)H(t) = \hat{U}_1^k(x,t) * \hat{G}_k(x,t),$$
$$\hat{\theta}(x,t) = \theta(x,t)H(t) = \hat{U}_2^k(x,t) * \hat{G}_k(x,t), \quad k = 1,2. \tag{8}$$

Where $\hat{U}_j^k(x,t)$ is the Green's tensor of equations (1), which describes thermoelastic waves generated by concentrated pulse sources. Everywhere over the same indices in the product, the summation is from 1 to 2 (tensor convolution).

As is known in the theory of generalized functions, the solution to differential equations is unique in the convolution algebra with the Green's tensor of these equations.

Green's tensor of the equations of unbound thermoelasticity

Green's tensor $U_i^j(x,t)$ is a matrix of fundamental solutions of system (1) under the action of a pulsed concentrated force and a heat source of the form:

$$F_1 = \delta(x)\delta(t)\delta_1^j, \quad F_2 = \delta(x)\delta(t)\delta_2^j, \quad j = 1,2 \tag{9}$$

where δ_i^j is the Kronecker symbol. When j =1, a power source operates. At j =2 a heat source operates. The Green's tensor satisfies the following *radiation conditions*:

$$U_i^j(x,t) = 0, \quad t < 0,$$
$$U_i^j(x,t) \to 0, \quad t \to \infty \text{ для } \forall x \in R^1. \tag{10}$$
$$U_i^j(x,t) \to 0, \quad |x| \to \infty \text{ для } \forall t > 0$$

The Green's tensor was constructed in [14], it has the following form:

$$U_1^j(x,t) = \delta_1^j k^{-1}\frac{\partial \Sigma_1}{\partial t} - \delta_1^j \frac{\partial^2 \Sigma_1}{\partial x^2} - \delta_2^j \tilde{\gamma}\frac{\partial \Sigma_1}{\partial x} - \delta_1^j \Sigma_3(t)\delta(x) + \delta_2^j \tilde{\gamma}\frac{\partial \Sigma_2}{\partial x},$$
$$U_2^1 = 0, \qquad U_2^2 = c^2 \Sigma_3(x,t) + c^2 \frac{\partial^2 \Sigma_2}{\partial x^2} - \frac{\partial^2 \Sigma_2}{\partial t^2}, \quad j = 1;2 \tag{11}$$

$$\Sigma_1(x,t) = \frac{k}{c} H(ct - |x|)\left[\frac{k}{c^2}\left(1 - e^{\frac{c}{k}(|x| - ct)}\right) + t - \frac{|x|}{c}\right],$$

$$\Sigma_2(x,t) = AH(t)\left\{e^{-\frac{c^2}{k}t}\int_0^t \frac{e^{\frac{c^2}{k}\tau - \frac{x^2}{4k\tau}}}{\sqrt{\tau}}d\tau - \int_0^t \frac{e^{-\frac{x^2}{4k\tau}}}{\sqrt{\tau}}d\tau\right\}, \ \Sigma_3(x,t) = -\frac{k}{c^2}H(t)(e^{-\frac{c^2}{k}t} - 1),$$

$$\frac{\partial \Sigma_1}{\partial t} = -\frac{k}{c}H(ct - |x|)\left[e^{\frac{c}{k}(|x| - ct)} - 1\right], \ \frac{\partial \Sigma_1}{\partial x} = \frac{k}{c^2}H(ct - |x|)\operatorname{sgn} x\left[e^{\frac{c}{k}(|x| - ct)} + 1\right],$$

$$\frac{\partial \Sigma_2}{\partial x} = -\frac{AH(t)}{k}\left\{e^{-\frac{c^2}{k}t}\int_0^t \frac{e^{\frac{c^2}{k}\tau - \frac{x^2}{4k\tau}}}{\sqrt{\tau}}\frac{x}{2\tau}d\tau - \int_0^t \frac{e^{-\frac{x^2}{4k\tau}}}{\sqrt{\tau}}\frac{x}{2\tau}d\tau\right\},$$

$$\frac{\partial^2 \Sigma_1}{\partial x^2} = \left(-\frac{k}{c^2}\delta(ct - |x|) + \frac{2k}{c^2}H(ct - |x|)\delta(x)\right)\left[e^{\frac{c}{k}(|x| - ct)} + 1\right] + \frac{1}{c}H(ct - |x|)e^{\frac{c}{k}(|x| - ct)},$$

$$\frac{\partial^2 \Sigma_2}{\partial t^2} = AH(t)\left\{\frac{c^4}{k^2} e^{-\frac{c^2}{k}t} \int_0^t \frac{e^{\frac{c^2}{k}\tau - \frac{x^2}{4k\tau}}}{\sqrt{\tau}} d\tau - \frac{c^2}{k} e^{-\frac{c^2}{k}t} \frac{e^{\frac{c^2}{k}\tau - \frac{x^2}{4k\tau}}}{\sqrt{\tau}}\right\} -$$

$$-AH(t)\left\{e^{-\frac{c^2}{k}t} \frac{e^{\frac{c^2}{k}t - \frac{x^2}{4kt}}}{k\sqrt{t}}\left(c^2 - \frac{x^2}{4t^2}\right) + \frac{x^2}{4kt^2} \frac{e^{-\frac{x^2}{4kt}}}{\sqrt{t}}\right\}.$$

$$\frac{\partial^2 \Sigma_2}{\partial x^2} = \frac{AH(t)}{2k}\left\{e^{-\frac{c^2}{k}t}\left[\int_0^t \frac{e^{\frac{c^2}{k}\tau - \frac{x^2}{4k\tau}}}{\tau\sqrt{\tau}}\left(\frac{x^2}{2k\tau} - 1\right) d\tau\right] - \int_0^t \frac{e^{-\frac{x^2}{4k\tau}}}{\tau\sqrt{\tau}}\left(\frac{x^2}{2k\tau} - 1\right) d\tau\right\}.$$

With knowledge of the Green's tensor, it becomes feasible to formulate a solution to system (7) for any sources using a tensor-functional convolution (8).

In equation (8), the convolution is performed componentwise in accordance with the definition of convolution in the space of generalized functions [12,13]. For regular sources described by locally integrable functions, this formula can be expressed in integral form

$$u_j(x,t) = \iint_{R^2} U_j^k(x-y, t-\tau) * F_k(y,\tau) dy d\tau \quad j,k = 1,2. \tag{12}$$

Solution of the Cauchy problem for determining the thermal stress state of a thermoelastic rod under the action of distributed force and heat sources. The method of generalized functions was used for the study.

Solutions to the Cauchy problem

To obtain an integral representation of the generalized solution, we take convolutions (8) taking into account (7), using the properties δ of the functions and its derivative (variable x under the convolution sign means incomplete convolution with respect to *x only*):

$$\hat{u}(x,t) = u(x,t)H(t) = \hat{U}_1^1(x,t) * F_1(x,t)H(t) + \hat{U}_1^1(x,t) \underset{x}{*} \upsilon_0(x) +$$
$$+ \frac{\partial}{\partial t}\hat{U}_1^1(x,t) \underset{x}{*} u_0(x) + \hat{U}_1^2(x,t) \underset{x}{*} F_2(x,t)H(t) + \hat{U}_1^2(x,t) \underset{x}{*} \kappa^{-1}\theta_0(x) \tag{13}$$

$$\hat{\theta}(x,t) = \theta(x,t)H(t) = \hat{U}_2^2(x,t) * F_2(x,t)H(t) + \hat{U}_2^2(x,t) \underset{x}{*} \kappa^{-1}\theta_0(x) \tag{14}$$

Note that all initial conditions of the Cauchy problem are included in the right-hand side of relations (13), (14). Since the functions included in the right side of equations (13), (14) are regular, these representations of the solution can be written in the following integral form.

Let us formulate the following theorem.

Theorem. If the functions $F_1(x,t), F_2(x,t)$ *are integrable on* R^2 *and* $u_0(x), v_0(x), \theta_0(x)$ *are continuous on R, then the solution to the Cauchy problem for the equations of uncoupled thermoelasticity has the form:*

$$\hat{u}(x,t) = H(t)\int_0^t d\tau \int_{-\infty}^{\infty} U_1^1(x-y,\tau)F_1(y,t-\tau)dy + H(t)\int_{-\infty}^{\infty} U_1^1(x-y,\tau)\upsilon_0(y)dy +$$
$$+H(t)\frac{\partial}{\partial t}\int_{-\infty}^{\infty} U_1^1(x-y,\tau)u_0(y)dy + H(t)\int_{-\infty}^{\infty} U_1^2(x-y,\tau)F_2(y,t-\tau)dy + \tag{15}$$
$$+\kappa^{-1}\int_{-\infty}^{\infty} U_1^2(x-y,\tau)\theta_0(y)dy,$$

$$\hat{\theta}(x,t) = H(t)\int_0^t d\tau \int_{-\infty}^{\infty} U_2^2(x-y,\tau)F_2(y,t-\tau)dy + \kappa^{-1}H(t)\int_{-\infty}^{\infty} U_2^2(x-y,\tau)\theta_0(y)dy. \qquad (16)$$

Using the Green's tensor representation (11) and $\hat{G}_1(x,t), \hat{G}_2(x,t)$ (7), the solution to the Cauchy problem has the form :

$$u(x,t) = \hat{U}_1^j * \hat{G}_j =$$

$$= \delta_1^j \kappa^{-1} \frac{\partial \Sigma_1}{\partial t} * G_j - \delta_1^j \frac{\partial^2 \Sigma_1}{\partial x^2} * G_j - \delta_2^j \tilde{\gamma} \frac{\partial \Sigma_1}{\partial x} * G_j - \delta_1^j \Sigma_3(t)\delta(x) * G_j + \delta_2^j \tilde{\gamma} \frac{\partial \Sigma_2}{\partial x} * G_j =$$

$$= \left\{ \left(\kappa^{-1} \frac{\partial \Sigma_1}{\partial t} - \frac{\partial^2 \Sigma_1}{\partial x^2} \right) - \Sigma_3(t) \right\} * G_1 + \tilde{\gamma} \left(\frac{\partial \Sigma_2}{\partial x} - \frac{\partial \Sigma_1}{\partial x} \right) * G_2 =$$

$$= \left\{ \left(\kappa^{-1} \frac{\partial \Sigma_1}{\partial t} - \frac{\partial^2 \Sigma_1}{\partial x^2} \right) - \Sigma_3(t) \right\} * \hat{F}_1 + \tilde{\gamma} \left(\frac{\partial \Sigma_2}{\partial x} - \frac{\partial \Sigma_1}{\partial x} \right) * \hat{F}_2 +$$

$$\left\{ \left(\kappa^{-1} \frac{\partial \Sigma_1}{\partial t} - \frac{\partial^2 \Sigma_1}{\partial x^2} \right) - \Sigma_3(t) \right\} \underset{x}{*} v_0(x) - \kappa^{-1}\tilde{\gamma} \left(\frac{\partial \Sigma_2}{\partial x} - \frac{\partial \Sigma_1}{\partial x} \right) \underset{x}{*} \theta_0(x) +$$

$$+ \partial_t \left\{ \left(\kappa^{-1} \frac{\partial \Sigma_1}{\partial t} - \frac{\partial^2 \Sigma_1}{\partial x^2} \right) - \Sigma_3(t) \right\} \underset{x}{*} u_0(x),$$

$$\theta(x,t) = \hat{U}_2^j * \hat{G}_j = c^2 \Sigma_3(x,t) * G_2 + c^2 \frac{\partial^2 \Sigma_2}{\partial x^2} * G_2 - \frac{\partial^2 \Sigma_2}{\partial t^2} * G_2 =$$

$$= \left(c^2 \Sigma_3(x,t) + c^2 \frac{\partial^2 \Sigma_2}{\partial x^2} - \frac{\partial^2 \Sigma_2}{\partial t^2} \right) * F_2(x,t)H(t) - \left(c^2 \Sigma_3(x,t) + c^2 \frac{\partial^2 \Sigma_2}{\partial x^2} - \frac{\partial^2 \Sigma_2}{\partial t^2} \right) \underset{x}{*} \kappa^{-1}\theta_0(x) \qquad (17)$$

Let us introduce the notation:

$$f1(x,t) = \kappa^{-1} \frac{\partial \Sigma_1}{\partial t} - \frac{\partial^2 \Sigma_1}{\partial x^2} - \Sigma_3(t), \quad f2(x,t) = \frac{\partial \Sigma_2}{\partial x} - \frac{\partial \Sigma_1}{\partial x},$$

$$f3(x,t) = c^2 \Sigma_3(x,t) + c^2 \frac{\partial^2 \Sigma_2}{\partial x^2} - \frac{\partial^2 \Sigma_2}{\partial t^2}$$

Then the integral representation (15, 16) has the following form:

$$u(x,t) = H(t)\int_0^t d\tau \int_{-\infty}^{\infty} \left\{ f1(x-y,t-\tau)F_1(y,\tau) + \tilde{\gamma} f2(x-y,t-\tau) * F_2(y,\tau) \right\} dy +$$

$$+ H(t)\int_{-\infty}^{\infty} \left\{ f1(x-y,t)v_0(y) - \kappa^{-1}\tilde{\gamma} f2(x-y,t)\theta_0(y) + \partial_t f1(x-y,t)u_0(y) \right\} dy$$

$$\theta(x,t) = H(t)\int_0^t d\tau \int_{-\infty}^{\infty} f3(x-y,t-\tau)F_2(y,\tau)dy - \kappa^{-1}H(t)\int_0^t f3(x-y,t)\theta_0(y)dy$$

Note that formulas (15,16) can also be used under the action of singular sources $\hat{F}_1(x,t)$, $\hat{F}_2(x,t)$, only in this case the convolutions must be taken according to the rules of convolutions in the space of generalized functions [12,13]. It's worth noting that formulas (15) and (16) remain applicable even when singular sources $\hat{F}_1\ (x,t)$, $\hat{F}_2\ (x,t)$ are present. However, in such cases, convolutions must be calculated here to the rules governing convolutions in the space of generalized functions [12,13].

Results of the study

The result of the conducted research is solutions to the Cauchy problem under the action of distributed power and heat sources, numerical calculations of the temperature and displacement of the rod under the action of a pulsed concentrated force and heat sources.

Numerical calculations of the Green's tensor

A program for computing the Green's tensor $U_i^j(x,t)$ has been developed in the Mathcad 15 system. Below (Figure 1) there are the calculations of the components of this matrix for the following dimensionless coordinates thermoelastic parameter: $\rho = 1,\quad c = 1,\quad \gamma = 1,\quad \kappa = 2$.

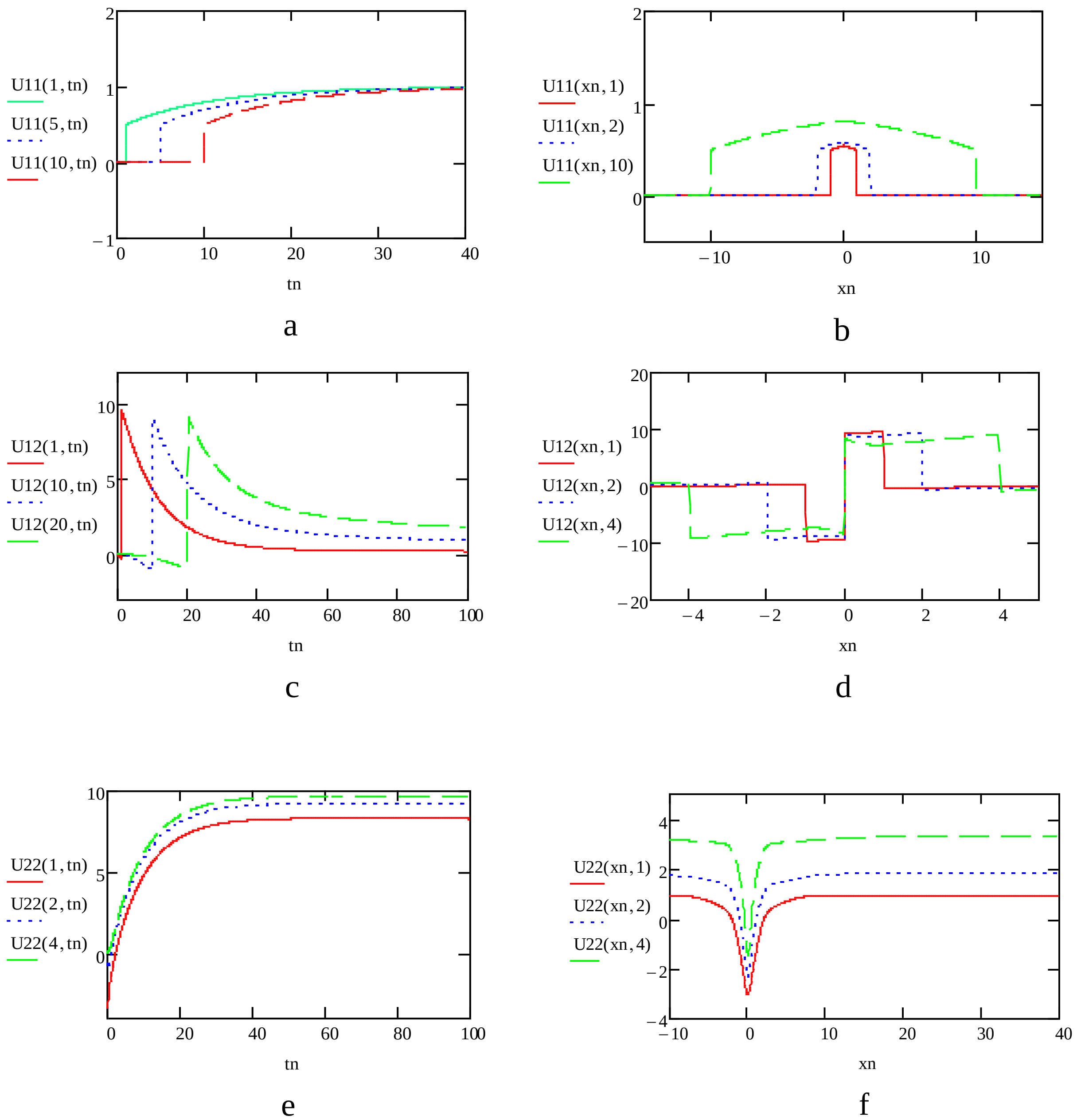

Figure 1(a-f). Green's tensor components $\mathrm{Uij}(xn,tn) = U_i^j(x,t)$ *in a fixed point and at a fixed time*

Figures 1 *(a-f)* depict the evolution over time of the respective components of the Green's tensor at a specific point in the medium *(x=xn)* and at a fixed time point *(t=tn)*. The discontinuities observed in the displacement graphs correspond to the arrival of the elastic shock wave at point *xn* at time *t=xn/c*, *t=xn/c*.

Solutions to the Cauchy problem under the action of distributed power and heat sources

Let us construct solutions to the Cauchy problem under the action of various power and heat sources of a regular form.

Cauchy problem 1. Let the following force be applied to a segment [-*L*,*L*] of the rod:

$$F_1(x,t) = A(x,t)H(L-|x|),\ F_2(x,t) = 0\ .$$

The initial conditions are zero. Then , using formulas (15), (16), Theorem 1, we obtain the displacement:

$$u(x,t) = U_1^1(x,t) * A(x,t)H(L-|x|) = H(t)\int_0^t d\tau \int_{-L}^{L} U_1^1(x-y,\tau)A(y,t-\tau)d\tau\ .$$

Next follows from (2):

$$\sigma_1(x,t) = \rho c^2 u,_x = \rho c^2 U_{1,x}^1(x,t) * F_1(x,t) = \rho c^2 U_1^1(x,t) * \frac{\partial F_1(x,t)}{\partial x} =$$

$$= \rho c^2 U_1^1(x,t) * (A(x,t)H(L-|x|)),_x =$$

$$= \rho c^2 U_1^1(x,t) * A(x,t)\left\{\delta(L+x) - \delta(L-x)\right\} + \rho c^2 U_1^1(x,t) * (A,_x H(L-|x|)) =$$

$$= \rho c^2 H(t)\int_0^t \left\{U_1^1(x+L,\tau)A(-L,t-\tau) - U_1^1(x-L,\tau)A(L,t-\tau)\right\}d\tau$$

$$+\rho c^2 H(t)\int_{-L}^{L} dy \int_0^t U_1^1(x-y,\tau)A(y,t-\tau)d\tau.$$

Cauchy problem 2. Let a heat source act on the segment [-*L*,*L*]:

$$F_1(x,t) = 0,\ F_2(x,t) = A(x,t)H(L-|x|)\ .$$

The initial conditions are zero. Then

$$u(x,t) = U_1^2(x,t) * A(x,t)H(L-|x|) = H(t)\int_0^t d\tau \int_{-L}^{L} U_1^2(x-y,\tau)A(y,t-\tau)dy,$$

$$\theta(x,t) = U_2^2(x,t) * A(x,t)H(L-|x|) = H(t)\int_0^t d\tau \int_{-L}^{L} U_2^2(x-y,\tau)A(y,t-\tau)dy\ ,$$

$$\sigma(x,t) = \rho c^2 u,_x - \gamma\theta =$$

$$= \rho c^2 H(t)\int_0^t \left\{U_1^2(x+L,\tau)A(-L,t-\tau) - U_1^2(x-L,\tau)A(L,t-\tau)\right\}d\tau +$$

$$+\rho c^2 H(t)\int_{-L}^{L} dy \int_0^t \left\{U_1^2(x-y,\tau) - \gamma U_2^2(x-y,\tau)\right\} A(y,t-\tau))d\tau.$$

Cauchy problem 3. Initial conditions are zero. Let a force be applied on a segment of the rod: $F_1(x,t) = A(x)H(L-|x|)H(t),\ F_2(x,t) = 0$. This solution:

$$u(x,t) = U_1^1(x,t) * A(x,t)H(L-|x|) = H(t)\int_{-L}^{L} A(x-y)dy \int_0^t U_1^1(y,\tau)d\tau,$$

$$\sigma(x,t) = \rho c^2 H(t)\int_0^t \left\{U_1^1(x+L,\tau)A(-L) - U_1^1(x-L,\tau)A(L)\right\}d\tau +$$

$$+\rho c^2 H(t)\int_{-L}^{L} A(x-y)dy\int_0^t U_1^1(y,\tau)d\tau$$

Cauchy problem 4. Let a heat source act on the segment [-*L*,*L*]: $F_1(x,t)=0,\ F_2(x,t)=A(x)H(t)H(L-|x|)$. The initial conditions are zero. In this case

$$u(x,t) = U_1^2(x,t) * A(x,t)H(L-|x|) = H(t)\int_0^t d\tau \int_{-L}^{L} U_1^2(x-y,\tau)A(y)dy,$$

$$\theta(x,t) = U_2^2(x,t) * A(x,t)H(L-|x|) = H(t)\int_0^t d\tau \int_{-L}^{L} U_2^2(x-y,\tau)A(y)dy$$

$$\sigma(x,t) = \rho c^2 u,_x - \gamma\theta =$$

$$= \rho c^2 H(t)\int_0^t \left\{U_1^2(x+L,\tau)A(-L) - U_1^2(x-L,\tau)A(L)\right\}d\tau +$$

$$+\rho c^2 H(t)\int_{-L}^{L} A(y)dy\int_0^t \left\{U_1^2(x-y,\tau) - \gamma U_2^2(x-y,\tau)\right\})d\tau.$$

With the joint distribution of power and heat sources of this type, the solution will be the sum of the corresponding solutions for each of them.

Discussion

In this study, a generalized solution to the Cauchy problem for a thermoelastic rod subjected to nonstationary force loads and heat sources has been constructed using the model of uncoupled thermoelasticity and the method of generalized functions. This approach has proven effective for addressing the complexity of initial-boundary value problems arising in nonstationary thermal environments, where classical methods may face difficulties due to the presence of singularities and discontinuities. The main advantage of the proposed method lies in its ability to handle singularities analytically by introducing generalized functions such as simple and double layers. These functions rigorously incorporate the effects of initial temperature, displacements, and velocity conditions into the mathematical model. The use of the Green’s tensor, previously developed in [9], allows for a systematic representation of the solution in both singular and regular (integral) forms, making it suitable for further numerical implementation. The numerical realization of the Green's tensor in Mathcad 15 for dimensionless thermoelastic parameters has shown that this approach is not only theoretically sound but also practically applicable. The solutions obtained for distributed force and thermal loads along the rod demonstrate how transient thermal effects interact with mechanical responses, particularly in cases involving sharp temperature gradients and time-dependent boundary conditions. These results contribute significantly to the understanding of the stress-strain behavior of rod structures under unsteady thermal loading. In engineering practice, such findings are critical for the design and safety assessment of components operating in thermally dynamic environments—such as turbine blades, reactor components, or high-speed mechanical systems—where repeated thermal and mechanical loads may otherwise lead to failure due to fatigue or thermal shock. Future work could extend the current model to coupled thermoelasticity, consider nonlinear material behavior, or include additional physical effects such as damping or viscoelasticity. Moreover, applying the method to more complex geometries and three-dimensional structures could further expand its applicability to real-world engineering systems.

Conclusion

The derived formulas and the implemented program enable the investigation of thermodynamic processes in rods subjected to diverse heat and power sources, whether distributed, pulsed, or concentrated, described by singular generalized functions. The outcomes of this study, along with the computer program, offer a means to evaluate the strength and reliability of rod structures throughout operation. These structures find applications in mechanical engineering as well as in the construction of above -ground and underground structures, including building supports, columns, and similar components.

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