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## SOLITONS IN THE COUPLED GENERALIZED NONLINEAR SCHRÖDINGER EQUATIONS

### Abstract

Nonlinear equations essentially describe physical problems. Solitons, which are localized in space and time and propagate in a nonlinear medium, are the special specific solutions to some of these equations. The soliton reemerges after a fully nonlinear interaction, retaining their identities with the same speed and shape. It has remarkable stability properties. Stability plays an important role in soliton physics. The nonlinear Schrödinger equation, one of the soliton-type nonlinear equations, is notable for its significance in the theory of nonlinear waves, specifically in nonlinear optics and plasma physics. The investigation of nonlinear Schrödinger equation generalizations is currently a pressing area of research. This work examines the coupled generalized nonlinear Schrödinger equations. These equations yield classical nonlinear Schrödinger equation in some reductions. As a research method, we use the bilinear Hirota method, which is one of the effective methods for solving nonlinear equations. Dispersion relations are derived and one-soliton and two-soliton solutions are obtained. The figures depict the dynamics of soliton solutions.

From this research, the results obtained may be useful for a better understanding of the nonlinear wave phenomena in any varied instance where the coupled model considered is applicable.

**Keywords:** Schrödinger equation, dispersion, dynamics, one soliton, two solitons, Hirota method, nonlinearity, solution, bilinearity, wave.

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## ЖҰПТАЛҒАН ЖАЛПЫЛАНҒАН СЫЗЫҚТЫ ЕМЕС ШРЕДИНГЕР ТЕНДЕУЛЕРІНІҢ СОЛИТОНДАРЫ

### Аңдатпа

Сызықты емес теңдеулер негізінен физикалық есептерді сипаттайды. Кеңістік пен уақытта локализацияланған және сызықты емес ортада таралатын солитондар осы теңдеулердің кейбірінің арнайы спецификалық шешімдері болып табылады. Солитон толық сызықты емес әрекеттесуден кейін қайта пайда болып, бірдей жылдамдықпен және пішінмен сәйкестіктерін сақтайды. Ол керемет тұрақтылық қасиеттеріне ие. Тұрақтылық солитондар физикасында маңызды рөл атқарады. Солитон типті сызықты емес теңдеулердің бірі болып табылатын сызықты емес Шредингер теңдеуі сызықты емес толқындар теориясында, әсіресе сызықты емес оптикада және плазма физикасында маңыздылығымен ерекшеленеді. Сызықтық емес Шредингердің теңдеулерін жалпылауды зерттеу қазіргі уақытта зерттеудің өзекті саласы болып табылады. Бұл жұмыс жұпталған жалпыланған сызықты емес Шредингер теңдеулерін зерттейді. Бұл теңдеулер кейбір қысқартуларда классикалық сызықты емес Шредингер теңдеуін береді. Зерттеу әдісі ретінде сызықты емес теңдеулерді шешудің тиімді әдістерінің бірі болып табылатын бисызықты Хирота әдісін қолданамыз. Дисперсиялық қатынастар алынып, бір солитонды және екі солитонды шешімдер табылады. Суреттер солитон шешімдерінің динамикасын бейнелейді. Осы зерттеуде алынған нәтижелер қарастырылатын жұпталған модель қолданылатын кез келген әртүрлі жағдайларда сызықтық емес толқын құбылыстарын жақсы түсіну үшін пайдалы болуы мүмкін.

**Түйін сөздер:** Шредингер теңдеуі, дисперсия, динамика, бір солитон, екі солитон, Хирота әдісі, сызықтық емес, шешім, бисызықтық, толқын.

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## СОЛИТОНЫ В СВЯЗАННЫХ ОБОБЩЕННЫХ НЕЛИНЕЙНЫХ УРАВНЕНИЙ ШРЕДИНГЕРА

### Аннотация

Нелинейные уравнения, по сути, описывают физические проблемы. Солитоны, локализованные в пространстве и времени и распространяющиеся в нелинейной среде, являются особыми специфическими решениями некоторых из этих уравнений. Солитон восстанавливается полностью после нелинейного взаимодействия, сохраняя свою идентичность с той же скоростью и формой. Он обладает замечательными свойствами устойчивости. Устойчивость играет важную роль в физике солитонов. Нелинейное уравнение Шредингера, одно из нелинейных уравнений солитонного типа, примечательно своей значимостью в теории нелинейных волн, в частности в нелинейной оптике и физике плазмы. Исследование обобщений нелинейного уравнения Шредингера в настоящее время является актуальной областью исследований. В этой работе рассматриваются связанные обобщенные нелинейные уравнения Шредингера. Эти уравнения дают классическое нелинейное уравнение Шредингера в некоторых редукциях. В качестве метода исследования применен билинейный метод Хироты, который является одним из эффективных методов решения нелинейных уравнений. Выводятся дисперсионные соотношения и получены односолитонные и двухсолитонные решения. На рисунках представлена динамика солитонных решений. Полученные в ходе данного исследования результаты могут быть полезны для лучшего понимания нелинейных волновых явлений в любых различных случаях, где применима рассматриваемая связанная модель.

**Ключевые слова:** уравнение Шредингера, дисперсия, динамика, один солитон, два солитона, метод Хироты, нелинейность, решение, билинейность, волна.

### Introduction

Almost every model of mathematical physics is the result of some approximate description of real physical phenomena. The purpose of approximations is to study the main interactions that determine the main contribution to the physics of the process, as well as to clarify the fundamental connections between phenomena. There are numerous physically valuable nonlinear equations, as the Korteweg-de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation [1-8]. In the last decade, a significant amount of work has been done on nonlinear equations [9-15].

In this work, we consider the coupled generalized nonlinear Schrodinger equations [16-18]

$$iq_t + q_{xx} + 2(a|p|^2 + c|q|^2 + bp\bar{q} + \bar{b}p\bar{q})q = 0, \quad (1)$$

$$ip_t + p_{xx} + 2(a|p|^2 + c|q|^2 + bp\bar{q} + \bar{b}p\bar{q})p = 0, \quad (2)$$

where  $a$  and  $c$  are real constants,  $b$  is a complex constant,  $q$  and  $p$  are slowly varying pulse envelopes. In case,  $b = 0, a = c$  then (1)-(2) reduce to the Manakov system [19]. And also, if  $b = 0, a = -c$  then (1)-(2) reduce to the mixed coupled nonlinear Schrodinger equation [20]. The equations (1)-(2) were investigated by Darboux transformation in [16].

The main goal of this work is to get soliton solutions of the coupled generalized nonlinear Schrodinger equations by Hirota's bilinear method [14,15]. A solitary wave with structural stability that travels over a nonlinear medium is called a soliton. Solitons behave like particles (particle-like waves) in that they do not break down when they interact with one another or with external disturbances; instead, they keep moving while maintaining their structural integrity. With the help of this property, data can be sent without interference over great distances. In addition, unlike harmonic waves, classical solitons, in addition to energy transfer, also transfer matter (shift in the direction of their movement over a finite distance).

**Research methodology**

*Bilinear form*

Numerous nonlinear equations have been solved using the Hirota method, a direct analytical technique for obtaining the soliton solutions. Once the bilinear form of the nonlinear equations was given, multi soliton solutions can be derived through the truncated formal perturbation expression at different levels. In this section, we apply the method to obtain soliton solutions for equations (1)-(2).

Introducing the dependent variable transformation

$$q = \frac{g}{f},$$

$$p = \frac{m}{f},$$

into (1)-(2), the bilinear form can be derived as follows:

$$[iD_t + D_x^2](g \cdot f) = 0 \tag{3}$$

$$[iD_t + D_x^2](m \cdot f) = 0, \tag{4}$$

$$D_x^2(f \cdot f) - 2(a|m|^2 + c|g|^2 + bm\bar{g} + \bar{b}g\bar{m}) = 0, \tag{5}$$

where  $g, m$  are complex functions,  $f$  is the real one and  $D_x$  and  $D_t$  are the bilinear differential operators defined by

$$D_x^l D_t^n (f(x, t) \cdot g(x, t)) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n (f(x, t) \cdot g(x', t'))|_{x'=x, t'=t}, \tag{6}$$

where  $l, n$  are positive integers.

*Soliton solution*

To get the N-solitons of the equations (1)-(2), we expand  $g, m$  and  $f$  in respect of a formal expansion with the parameter  $\varepsilon$  as below:

$$g = \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5 + \dots, \tag{7a}$$

$$m = \varepsilon m_1 + \varepsilon^3 m_3 + \varepsilon^5 m_5 + \dots, \tag{7b}$$

$$f = 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \dots, \tag{7c}$$

where  $g_j$ 's and  $m_j$ 's ( $j = 1, 3, 5, \dots$ ) are the complex functions, and  $f_n$ 's ( $n = 2, 4, 6, \dots$ ) are the real ones, to be determined. Substituting expression (7) into (3)-(5) and gathering the coefficients of the same power of  $\varepsilon$ , we have

$$\varepsilon^1: [iD_t + D_x](g_1 \cdot 1) = 0, \tag{8a}$$

$$\varepsilon^1: [iD_t + D_x](m_1 \cdot 1) = 0, \tag{8b}$$

$$\varepsilon^2: D_x^2(f_2 \cdot 1 + 1 \cdot f_2) - 2(a|m_1|^2 + c|g_1|^2 + bm_1\bar{g}_1 + \bar{b}g_1\bar{m}_1) = 0 \tag{8c}$$

$$\varepsilon^3: [iD_t + D_x](g_3 \cdot 1 + g_1 \cdot f_2) = 0, \tag{8d}$$

$$\varepsilon^3: [iD_t + D_x](m_3 \cdot 1 + m_1 \cdot f_2) = 0, \tag{8e}$$

$$\varepsilon^4: D_x^2(f_4 \cdot 1 + f_2 \cdot f_2 + 1 \cdot f_4) - 2(a(m_3\bar{m}_1 + m_1\bar{m}_3) + c(g_3\bar{g}_1 + g_1\bar{g}_3) + b(m_3\bar{g}_1 + m_1\bar{g}_3) + \bar{b}(g_3\bar{m}_1 + g_1\bar{m}_3)) = 0 \tag{8f}$$

...

Equation (1)-(2) has one-, two-, and N-soliton solutions, which we may find by using expression (8) and symbolic computation.

### Results of the study

#### One soliton solution

To get the one soliton for equations (1)-(2), we choose

$$g = \varepsilon g_1, m = \varepsilon m_1, f = 1 + \varepsilon^2 f_2. \tag{9}$$

Substituting expressions (9) into bilinear forms (3)-(5), setting  $\varepsilon = 1$ , we solve the bilinear system recursively and obtain one-soliton solution as

$$q = \frac{\alpha e^{\theta_1}}{1 + e^{\theta_1 + \bar{\theta}_1 + \varphi}}, \tag{10}$$

$$p = \frac{\beta e^{\theta_1}}{1 + e^{\theta_1 + \bar{\theta}_1 + \varphi}}, \tag{11}$$

where  $e^\varphi = \frac{a|\beta|^2 + c|\alpha|^2 + b\beta\bar{\alpha} + \bar{b}\alpha\bar{\beta}}{(k_1 + \bar{k}_1)^2}$  and  $\theta_1 = c_1 t + k_1 x$  with dispersion relation  $c_1 = ik_1^2$ .

Figures 1-3 illustrate the graphs of the one-soliton solutions (10)– (11).

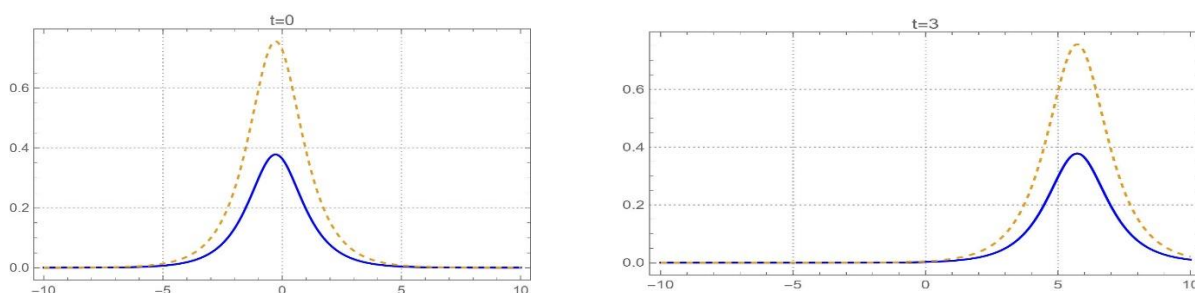


Figure 1. Dynamics of the one-soliton solutions'  $q$  and  $p$  with parameters  $k_1 = 1 + i; a = 1; b = 1; c = 1; \alpha = 1; \beta = 2$

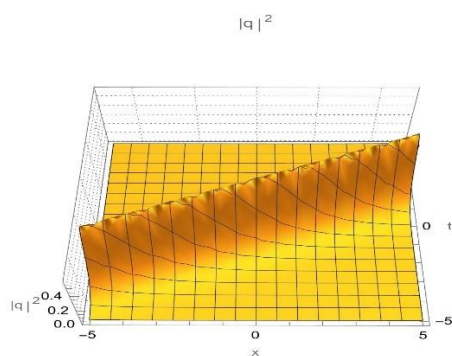


Figure 2. 3D plot of the two-soliton solutions  $q$  with parameters  $k_1 = 1 + i; a = 1; b = 1; c = 1; \alpha = 1; \beta = 2$

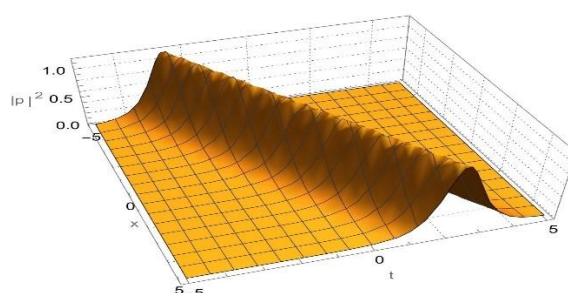


Figure 3. 3D plot of the two-soliton solutions  $q$  with parameters  $k_1 = 1 + i; a = 1; b = 1; c = 1; \alpha = 1; \beta = 2$

#### Two soliton solution

As in the previous section, we truncate  $g$ ,  $m$  and  $f$  perturbation expression with respect to the parameter  $\varepsilon$  as

$$g = \varepsilon g_1 + \varepsilon^3 g_3, m = \varepsilon m_1 + \varepsilon^3 m_3, f = 1 + \varepsilon^2 f_2 + \varepsilon^2 f_4. \tag{12}$$

Substituting expressions (12) into bilinear forms (3)-(5), solving the bilinear system recursively, and, setting  $\varepsilon = 1$ , the two-soliton solution can be derived as follows:

$$q = \frac{\alpha_1 e^{\theta_1 + \alpha_2} e^{\theta_1 + e^{\theta_1 + \theta_2 + \bar{\theta}_1 + \delta_1 + e^{\theta_1 + \theta_2 + \bar{\theta}_2 + \delta_2}}}{1 + e^{\theta_1 + \bar{\theta}_1 + \varphi_1 + e^{\theta_2 + \bar{\theta}_1 + \varphi_2 + e^{\theta_1 + \bar{\theta}_2 + \varphi_3 + e^{\theta_2 + \bar{\theta}_2 + \varphi_4 + e^{\theta_1 + \bar{\theta}_1 + \theta_2 + \bar{\theta}_2 + s}}}} \quad (13)$$

$$p = \frac{\beta_1 e^{\theta_1 + \beta_2} e^{\theta_1 + e^{\theta_1 + \theta_2 + \bar{\theta}_1 + L_1 + e^{\theta_1 + \theta_2 + \bar{\theta}_2 + L_2}}}{1 + e^{\theta_1 + \bar{\theta}_1 + \varphi_1 + e^{\theta_2 + \bar{\theta}_1 + \varphi_2 + e^{\theta_1 + \bar{\theta}_2 + \varphi_3 + e^{\theta_2 + \bar{\theta}_2 + \varphi_4 + e^{\theta_1 + \bar{\theta}_1 + \theta_2 + \bar{\theta}_2 + s}}}} \quad (14)$$

were

$$\begin{aligned} e^{\varphi_1} &= \frac{a|\beta_1|^2 + c|\alpha_1|^2 + b\beta_1\bar{\alpha}_1 + \bar{b}\alpha_1\bar{\beta}_1}{(k_1 + \bar{k}_1)^2}, & e^{\varphi_2} &= \frac{a\bar{\beta}_1\beta_2 + c\alpha_2\bar{\alpha}_1 + b\bar{\alpha}_1\beta_2 + \bar{b}\alpha_2\bar{\beta}_1}{(k_2 + \bar{k}_1)^2}, \\ e^{\varphi_3} &= \frac{a\beta_1\bar{\beta}_2 + c\alpha_1\bar{\alpha}_2 + b\bar{\alpha}_2\beta_1 + \bar{b}\alpha_1\bar{\beta}_2}{(k_1 + \bar{k}_2)^2}, & e^{\varphi_4} &= \frac{a|\beta_2|^2 + c|\alpha_2|^2 + b\beta_2\bar{\alpha}_2 + \bar{b}\alpha_2\bar{\beta}_2}{(k_2 + \bar{k}_2)^2}, \\ e^{\delta_1} &= \frac{\alpha_1(i(c_1 - c_2 - \bar{c}_1) + (k_1 - k_2 - \bar{k}_1)^2)}{i(c_1 + c_2 + \bar{c}_1) + (k_1 + k_2 + \bar{k}_1)^2} e^{\varphi_2} + \frac{\alpha_2(i(c_2 - c_1 - \bar{c}_1) + (k_2 - k_1 - \bar{k}_1)^2)}{i(c_1 + c_2 + \bar{c}_1) + (k_1 + k_2 + \bar{k}_1)^2} e^{\varphi_1}, \\ e^{\delta_2} &= \frac{\alpha_1(i(c_1 - c_2 - \bar{c}_2) + (k_1 - k_2 - \bar{k}_2)^2)}{i(c_1 + c_2 + \bar{c}_2) + (k_1 + k_2 + \bar{k}_2)^2} e^{\varphi_4} + \frac{\alpha_2(i(c_2 - c_1 - \bar{c}_2) + (k_2 - k_1 - \bar{k}_2)^2)}{i(c_1 + c_2 + \bar{c}_2) + (k_1 + k_2 + \bar{k}_2)^2} e^{\varphi_3}, \\ e^{L_1} &= \frac{\beta_1(i(c_1 - c_2 - \bar{c}_1) + (k_1 - k_2 - \bar{k}_1)^2)}{i(c_1 + c_2 + \bar{c}_1) + (k_1 + k_2 + \bar{k}_1)^2} e^{\varphi_2} + \frac{\beta_2(i(c_2 - c_1 - \bar{c}_1) + (k_2 - k_1 - \bar{k}_1)^2)}{i(c_1 + c_2 + \bar{c}_1) + (k_1 + k_2 + \bar{k}_1)^2} e^{\varphi_1}, \\ e^{L_2} &= \frac{\beta_1(i(c_1 - c_2 - \bar{c}_2) + (k_1 - k_2 - \bar{k}_2)^2)}{i(c_1 + c_2 + \bar{c}_2) + (k_1 + k_2 + \bar{k}_2)^2} e^{\varphi_4} + \frac{\beta_2(i(c_2 - c_1 - \bar{c}_2) + (k_2 - k_1 - \bar{k}_2)^2)}{i(c_1 + c_2 + \bar{c}_2) + (k_1 + k_2 + \bar{k}_2)^2} e^{\varphi_3}, \\ e^s &= + \frac{a(e^{L_2}\bar{\beta}_1 + e^{L_1}\bar{\beta}_2 + e^{L_2}\beta_1 + e^{L_1}\beta_2) + c(e^{\delta_2}\bar{\alpha}_1 + e^{\delta_1}\bar{\alpha}_2 + e^{\delta_2}\alpha_1 + e^{\delta_1}\alpha_2)}{(k_1 + k_2 + \bar{k}_1 + \bar{k}_2)^2} + \\ &+ \frac{b(e^{L_2}\bar{\alpha}_1 + e^{L_1}\bar{\alpha}_2 + e^{L_2}\alpha_1 + e^{L_1}\alpha_2) + \bar{b}(e^{\delta_2}\bar{\beta}_1 + e^{\delta_1}\bar{\beta}_2 + e^{\delta_2}\beta_1 + e^{\delta_1}\beta_2)}{(k_1 + k_2 + \bar{k}_1 + \bar{k}_2)^2} - \\ &- \frac{2e^{\varphi_1 + \varphi_4}(k_2 + k_1 - \bar{k}_1 - \bar{k}_2)^2 + 2e^{\varphi_1 + \varphi_4}(k_1 - k_2 - \bar{k}_1 + \bar{k}_2)^2}{(k_1 + k_2 + \bar{k}_1 + \bar{k}_2)^2}, \end{aligned}$$

with  $\theta_j = c_j t + k_j x$  and dispersion relations  $c_j = ik_j^2$  ( $j = 1, 2$ ).

Figures 4-6 present the graphs of the two-soliton solutions (13)-14).

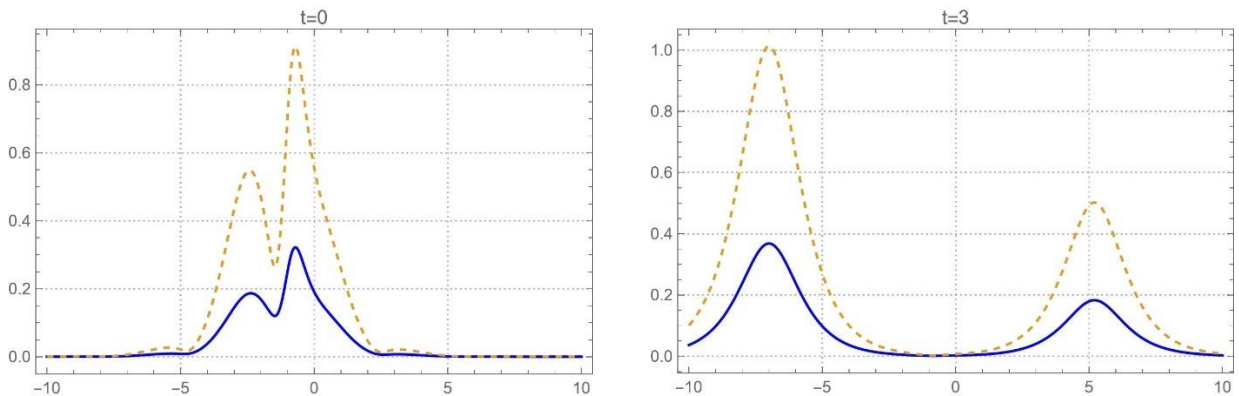


Figure 4. Dynamics of the two-soliton solutions  $q$  and  $p$  with parameters  $k_1 = 1 + i; k_2 = 1 - i; a = 1; b = 1; c = 1; \alpha_1 = 1; \beta_1 = 3.5; \alpha_2 = 2; \beta_2 = 5.5$ ,

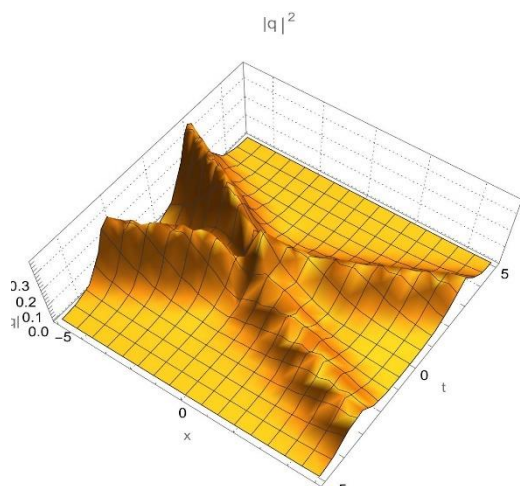


Figure 5. 3D plot of the two-soliton solutions  $q$  with parameters  
 $k_1 = 1 + i; k_2 = 1 - i; a = 1; b = 1; c = 1; \alpha_1 = 1; \beta_1 = 3.5; \alpha_2 = 2; \beta_2 = 5.5,$

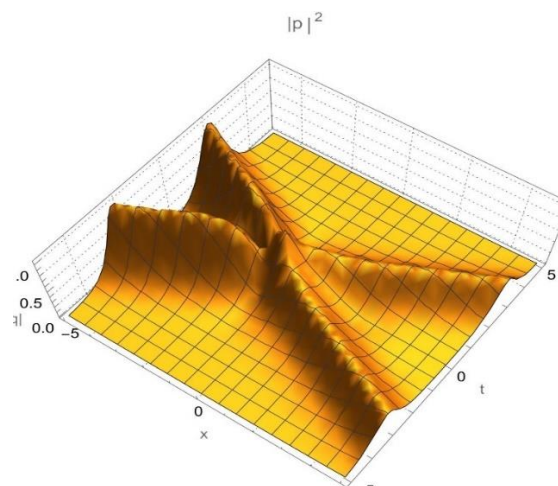


Figure 6. 3D plot of the two-soliton solutions  $p$  with parameters  
 $k_1 = 1 + i; k_2 = 1 - i; a = 1; b = 1; c = 1; \alpha_1 = 1; \beta_1 = 3.5; \alpha_2 = 2; \beta_2 = 5.5,$

## Conclusion

In this work, we investigated the coupled generalized nonlinear Schrodinger equations. This system has many applications in plasma physics and ocean waves. In some cases, the coupled generalized nonlinear Schrodinger equations can be reduced to nonlinear Schrodinger equations arising from various fields, such as quantum field and nonlinear optics.

To obtain a soliton solution we applied Hirota's method. This approach provides a direct method for finding N-soliton solutions to nonlinear dispersive equations. We got bilinear equations and constructed one soliton and two soliton solutions for the coupled generalized nonlinear Schrodinger equations. The dynamics of the soliton pulses have been presented in Figure 1 with the help of the Mathematica package software by selecting appropriate values for unknown parameters. Propagation of the optical solitons, generated via the balance between the group velocity dispersion and nonlinear effect, has potential applications in the nonlinear fibers. The presented methods can be applied to obtain new solutions for other nonlinear equations.

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