IRSTI 27.31.17

10.51889/2959-5894.2025.89.1.002

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MIXED-TYPE DIFFERENTIAL EQUATION WITH FRACTIONAL DERIVATIVE IN THE CAPUTO SENSE

Abstract

The purpose of this paper is to study boundary value problems for a mixed-type differential equation with fractional derivatives in the sense of Caputo and to demonstrate the existence and uniqueness of its solution. Such equations, including fractional derivatives, have significant potential to describe various physical processes in which the effects of memory and heredity are evident, such as abnormal diffusion, heat transfer, and relaxation phenomena. The paper presents an analytical approach to solving the problem based on the method of separating variables by representing the solution as a Fourier series. As a result, the conditions for the uniqueness of the solution were established and strictly proved, which, if certain conditions are met, ensure the ambiguity of the task. In addition, the uniform convergence of the obtained series of solutions is proved under the specified conditions. The results obtained can be used in the theory of differential equations and in further applied research.

Keywords: Caputo operator, uniqueness and existence of a solution, Fourier method, boundary value problem.

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КАПУТО МАҒЫНАСЫНДА БӨЛШЕК РЕТТІ ТУЫНДЫСЫ БАР АРАЛАС ТИПТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУ

Аңдатпа

Бұл жұмыстың мақсаты – Капуто мағынасындағы бөлшек ретті туындылары бар аралас типті дифференциалдық теңдеу үшін шекаралық есептерді зерттеу және оның шешімінің бар болуы мен жалғыздығын көрсету. Мұндай теңдеулер, соның ішінде бөлшек ретті туындылар, есте сақтау және тұқым қуалаушылық әсерлері көрінетін әртүрлі физикалық процестерді сипаттау үшін маңызды әлеуетке ие, мысалы, аномальды диффузия, жылу беру және релаксация құбылыстары. Жұмыста шешімді Фурье қатары түрінде көрсету арқылы айнымалыларды ажырату әдісіне негізделген мәселені шешудің аналитикалық тәсілі ұсынылған. Нәтижесінде белгілі бір шарттарды сақтай отырып, қойылған мәселенің анық еместігін қамтамасыз ететін шешімнің жалғыздығының шарттары белгіленді және қатаң дәлелденді. Сонымен қатар, белгіленген шарттар орындалған кезде алынған шешімдер қатарының бірқалыпты жинақтылығы дәлелденді. Алынған нәтижелерді дифференциалдық теңдеулер теориясында және әрі қарай қолданбалы зерттеулерде қолдануға болады.

Түйін сөздер: Капуто операторы, шешімнің жалғыздығы және бар болуы, Фурье әдісі, шекаралық есеп.

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ДИФФЕРЕНЦИАЛЬНОЕ УРАВНЕНИЕ СМЕШАННОГО ТИПА С ПРОИЗВОДНОЙ ДРОБНОГО ПОРЯДКА В СМЫСЛЕ КАПУТО

Аннотация

Целью данной работы является исследование краевых задач для дифференциального уравнения смешанного типа с дробными производными в смысле Капуто и демонстрация существования и единственности его решения. Такие уравнения, включая производные дробного порядка, обладают значительным потенциалом для описания различных физических процессов, в которых очевидны эффекты памяти и наследственности, таких как аномальная диффузия, перенос тепла и явления релаксации. В работе представлен аналитический подход к решению задачи, основанный на методе разделения переменных путем представления решения в виде ряда Фурье. В результате были установлены и строго доказаны условия единственности решения, которые при соблюдении определенных условий обеспечивают неоднозначность поставленной задачи. Кроме того, доказана равномерная сходимость полученного ряда решений при выполнении указанных условий. Полученные результаты могут быть использованы в теории дифференциальных уравнений и в дальнейших прикладных исследованиях.

Ключевые слова: оператор Капуто, единственность и существование решения, метод Фурье, краевая задача.

Main provisions

In the paper we consider a boundary value problem for a differential equation of mixed type with fractional derivatives in the sense of Caputo and obtains its analytical solution in the form of a Fourier series. Sufficient conditions for the existence and uniqueness of a solution are determined, and uniform convergence of the obtained series is proved. It is shown that the analytical method allows one to effectively take into account the memory effects common in physical systems. The results obtained open the way to studying multidimensional and nonlinear cases and will serve as a basis for future studies of the dependence of the solution on parameters using numerical modeling.

Introduction

In recent years, interest in boundary value problems for fractional differential equations has grown due to their application in memory-intensive modeling processes such as wave propagation in inhomogeneous media, plasma theory, and biophysics, as well as the growing need for accurate models to describe processes with inhomogeneous dynamics.

Equations with fractional differential operators such as Caputo derivatives are particularly useful in modeling complex dynamic systems. These equations are used in porous media, in problems of fluid filtration in electromagnetic waves, and in studying wave propagation processes in weakly dispersed and cold plasma media [1].

A significant amount of research is devoted to the study of boundary value problems for fractional differential equations of mixed type, where methods and conditions ensuring the uniqueness and existence of a solution are considered. In addition, the use of the Fourier method and other numerical approaches allows one to effectively solve such problems and obtain exact solutions for various physical applications, including magnetohydrodynamics and processes in space plasma. Differential equations of mixed type have been studied in the works of many authors, in particular [2-4].

Research methodology

This theoretical study was conducted in 2024 at the Faculty of Mathematics, Physics and Informatics of the Abai Kazakh National Pedagogical University. The aim of the study is to analyze boundary value problems for a mixed-type differential equation with fractional derivatives in the Caputo sense.

The study uses analytical methods of mathematical analysis. The solution to the boundary value problem was constructed by expanding it into a Fourier series with separated variables. This allows us to express the solution as a sum of orthogonal eigenfunctions. The properties of the Caputo fractional derivative are used to take into account the effects of fractional differentiation. The study is carried out in a model domain consisting of two subdomains: parabolic and hyperbolic. This division reflects the mixed nature of the equation under consideration. Theorems of existence and uniqueness of solutions are established on the basis of special conditions. A criterion for the uniqueness of a solution is formulated and proven. In this paper, we consider a boundary value problem for a differential equation with fractional derivatives in the Caputo sense.

Let $0 < \alpha \leq 1$. The f fractional derivative of f in the Caputo sense up to order α is as follows:

$$D_t^{\alpha} f(t) = \int_0^t h_{1-\alpha}(t-s)\frac{d}{ds}f(s)ds,\ h_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

Where $\Gamma(\alpha)$ is gamma function. Consider the following equation:

$$-u_{xx}(x,t) + D_t^{\alpha}u(x,t) = f(x),\ \ 0 < x < l,\ \ 0 < \mathrm{t} < \mathrm{T}. \tag{1}$$

For equation (1) there are boundary conditions:

$$u(0,t) = u(l,t) = 0,\ \ 0 < x < l, \tag{2}$$

and the initial conditions:

$$\varphi(x,0) = \varphi(x),\ \ \ 0 < x < l, \tag{3}$$

it is necessary to find a function $u(x,t)$ that satisfies equation. If $\alpha = 1$, then this problem corresponds to the usual classical problem.

Definition. The solution to problems (1)-(3) is the sum of all the conditions of the general problem and the following:

1) $D_t^{\alpha}u(x,t) \in C^2([0,l]\times[0,T])$;
2) $u_{xx}(x,t) \in C^2([0,l]\times[0,T])$;
3) $u(x,t) \in C^2([0,l]\times[0,T])$;

The function $u(x,t)$ is called satisfying the conditions. Here, $T > 0$ is a given real number.

We find the unknown function $u(x,t)$, which is a solution to problems (1)-(3), as follows [5]:

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x). \tag{4}$$

In the following spectral problem, we denote the system of orthonormal functions in the space $L^2(0,l)$ by $\{X_k\}$ and the set of positive eigenvalues by $\{\sqrt{\lambda_k}\}$ [7]:

$$\begin{cases} -\dfrac{\partial^2 X_k(x)}{\partial x^2} = \lambda_k X_k(x),\ \ 0 < x < l, \\ X(0) = X(l) = 0. \end{cases} \tag{5}$$

Accordingly, we obtain the following differential equation:

$$\begin{cases} D_t^\alpha T_k(t) + \lambda_k T_k(t) = f_k(t), \\ \varphi(0) = \varphi_k. \end{cases} \tag{6}$$

The solution to the differential equation is as follows [7]:

$$T_k(t) = \varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right) + \int_0^t \xi^{\alpha-1} f_k E_{\alpha,\alpha}\left(-\lambda_k \xi^\alpha\right) d\xi.$$

The unknown function $u(x,t)$, which is a solution to problems (1)-(3), has the following form:

$$u(x,t) = \sum_{k=1}^{\infty}\left[\varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right) + \int_0^t \xi^{\alpha-1} f_k E_{\alpha,\alpha}\left(-\lambda_k \xi^\alpha\right) d\xi\right]\sin\left(\sqrt{\lambda_k}x\right). \tag{7}$$

The existence and uniqueness of the solution

Theorem 1. If the problem has a solution, then it is unique.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be the solutions of problems (1)-(3). Let us prove that $u(x,t) = u_1(x,t) - u_2(x,t) \equiv 0$. To find the solution $u(x,t)$, we obtain the following homogeneous problem.

Problem statement.

Consider homogeneous equation

$$-\frac{\partial^2 u(x,t)}{\partial x^2} + D_t^\alpha u(x,t) = 0, \;\; 0 < x < l, \;\; 0 < \mathrm{t} < \mathrm{T}. \tag{8}$$

Find $u(x,t)$ a regular solution of equation (8), satisfying the boundary conditions (2) and initial conditions (3).

Let $u(x,t)$ satisfy all conditions of the homogeneous problem and let X_k be the eigenfunction corresponding to each eigenvalue $\sqrt{\lambda_k}$ of the spectral problem (5).

Consider the following function:

$$w_k(t) = \int_0^l T_k(t)\sin\left(\sqrt{\lambda_k}x\right)dx, \;\; k = 1,2,.... \tag{9}$$

Here, $T_k(t)$ depends on t but not on x, so it is taken from the integral notation. However, the original formula implies integration over x without eliminating the function $T_k(t)$. We differentiate (9) with the integral notation over time t:

$$\frac{d}{dt}w_k(t) = \frac{d}{dt}\int_0^l T_k(t)\sin\left(\sqrt{\lambda_k}x\right)dx = \int_0^l \frac{d}{dt}T_k(t)\sin\left(\sqrt{\lambda_k}x\right)dx.$$

We consider the original differential equation (8) and the function (4) classified by eigenfunctions. When substituting into the equation, each term must satisfy the equation, and as a result, we obtain the differential equation:

$$D_t^\alpha w_k(t) - \lambda_k w_k(t) = 0$$

The solution to the resulting differential equation is as follows [6, Б. 17, (1.3.3)]:

$$w_k(t) = A_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right), \ k = 1,2,..., \ 0 < t < T, \quad (10)$$

Since $\varphi(0) = 0$ by the conditions, $A_k = 0$. Then the right side of the equation is zero. This means that $u(x,t)$ is completely orthogonal to the system $\{X_k\}$. Therefore, $u_1(x,t) = u_2(x,t)$, $u(x,t) \equiv 0$, $0 < x < l$. The theorem is proved. ☐

We introduce some properties of the Mittag-Leffler function.

Lemma 1. For any $t \geq 0$ the following estimate is valid:

$$E_{\alpha,\mu}(-t) \leq \frac{C}{1+t}, \quad (13)$$

where C is a constant independent of t and μ.

Lemma 2. The classical Mittag-Leffler function of the negative argument of $E_\alpha(-t)$ is a monotonically decreasing function for all $0 < \alpha < 1$ and

$$0 < E_\alpha(-t) < 1, \ E_\alpha(0) = 1 \quad (14)$$

is feasible.

Lemma 3. Let $0 < \alpha < 1$ and $\lambda > 0$ be such that for all positive t the following holds [8]:

$$\int_0^t \xi^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda\xi^\alpha\right) d\xi = t^\alpha E_{\alpha,\alpha+1}\left(-\lambda t^\alpha\right). \quad (15)$$

Theorem 2. Let $f(x) \in C[0,l]$ be a continuously differentiable function. Then equations (1)-(3) have a solution $u(x,t)$ and are given by the series (6) uniformly and absolutely convergent on the interval $0 < x < l$ for all $t \in (0,T)$.

Proof. We differentiate the solution (6) twice with respect to

$$u_{xx}(x,t) = \sum_{k=1}^{\infty} -\lambda_k \left[\varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right) + \lambda_k \int_0^t \xi^{\alpha-1} f_k E_{\alpha,\alpha}\left(-\lambda_k \xi^\alpha\right) d\xi\right] \sin\left(\sqrt{\lambda_k} x\right).$$

Let's break this row into two parts:

$$\sum_{k=1}^{\infty} -\lambda_k \varphi_k E_{\alpha,1}\left(-\lambda_k t^\alpha\right), \quad (16)$$

$$\sum_{k=1}^{\infty} -\lambda_k \int_0^t \xi^{\alpha-1} f_k E_{\alpha,\alpha}\left(-\lambda_k t^\alpha\right) \mathrm{d}\xi. \quad (17)$$

If the function $\varphi(x)$ is four-times differentiable and the condition $\varphi(0) = \varphi(l) = 0$ is true, then the series (10) converges.

Using the estimate (13), we consider the modulus of the series (16).

$$\sum_{k=1}^{\infty}\lambda_k\left|\varphi_k E_{\alpha,1}\left(-\lambda_k t^{\alpha}\right)\right| \le \sum_{k=1}^{\infty}\lambda_k\left|\varphi_k\frac{1}{1+\lambda_k t^{\alpha}}\right| \le$$

$$\le \sum_{k=1}^{\infty}\lambda_k\frac{1}{\lambda_k t^{\alpha}}\left|\varphi_k\right| = \sum_{k=1}^{\infty}t^{-\alpha}\left|\varphi_k\right| = t^{-\alpha}\sum_{k=1}^{\infty}\left|\varphi_k\right|.$$

Here,

$$\varphi_k = \frac{2}{l}\int_0^l \varphi(x)\sin\left(\sqrt{\lambda_k}x\right)dx = \frac{2}{(\pi k)^2}\int_0^l \varphi^{(4)}(x)\sin\left(\sqrt{\lambda_k}x\right)dx.$$

Then the series (16) converges absolutely and uniformly.

By following the same steps, we investigate the convergence of the series (17).

If the function $f(x)$ is four-times differentiable and the condition $f(0)=f(l)=0$ is true, then the series (17) converges.

$$\sum_{k=1}^{\infty}\lambda_k f_k\left|t^{\alpha}\frac{1}{1+\lambda_k t^{\alpha}}\right| \le \sum_{k=1}^{\infty}f_k\frac{1}{t^{\alpha}}\lambda_k\left|\frac{1}{\lambda_k}\right| = \frac{1}{t^{\alpha}}\sum_{k=1}^{\infty}f_k.$$

Here,

$$f_k = \frac{2}{l}\int_0^l f(x)\sin\left(\sqrt{\lambda_k}x\right)dx = \frac{2}{(\pi k)^2}\int_0^l f^{(4)}(x)\sin\left(\sqrt{\lambda_k}x\right)dx$$

Then the series (17) converges absolutely and uniformly.

If $u_{xx}(x,t)\in C^2([0,l]\times[0,T])$ is absolutely and uniformly convergent, then $u(x,t)\in C^2([0,l]\times[0,T])$ converges.

According to equation (1):

$$D_t^{\alpha}u(x,t) = f(x) + u_{xx}(x,t)$$

Therefore, $D_t^{\alpha}u(x,t)\in C([0,l]\times[0,T])$ is absolutely and uniformly convergent.

Theorem 2 is proved. ☐

Results of the study

As a result of theoretical studies, sufficient conditions for the existence and uniqueness of a solution to a boundary value problem for a differential equation of mixed type with fractional derivatives in the sense of Caputo were established. The solution was given in the form of a Fourier series and its uniform convergence was proven when the formulated uniqueness conditions were met. The obtained results confirm the validity of the proposed hypothesis about the existence of a stable and unique solution in a limited region divided into hyperbolic and parabolic parts. An existence theorem was proven, and the uniqueness conditions showed its applicability to various sets of boundary conditions.

Discussion

As a result of this study, a boundary value problem for a differential equation of mixed type with fractional derivatives in the sense of Caputo is solved by a rigorous analytical method. Its solution is given in an exact form using a Fourier series, which allows one to express the solution clearly without using numerical methods. Below, the importance of this approach, its relation to memory effects in physical systems, and a comparison of this work with other modern studies using the work of Caputo are discussed, and future research directions are outlined. The analytical approach used in the study allows one to obtain an exact solution to the equation and provides a deeper understanding of the behavior of the system. Unlike numerical approximation methods, the analytical solution completely describes the dependence of the problem on the parameters and initial conditions, eliminating approximation errors. Representing the solution as a Fourier series not only clearly shows the influence of each harmonic component of the solution, but also ensures that the solution approaches the true value as the number of terms in the series increases. The significance of this approach is that it allows one to analyze the influence of computational parameters (e.g., the order of the fractional derivative) and can serve as a benchmark for assessing the accuracy of approximate methods.

One of the unique features of this study is that the problem is solved purely analytically, without using numerical methods. Usually, fractional equations cannot be solved by simple analytical formulas, which forces us to resort to numerical methods. Therefore, the analytical solution in our work has significant value in this context. The analytical solution completely describes the behavior of the system and can serve as a basis for checking the correctness of other numerical results. This approach also allows one to study the boundary conditions of the solution.

Caputo's work is currently used in many studies. A special feature of our work is that we obtain an exact analytical solution for a mixed-type equation. In other studies, the solution is often specified by special functions and only general properties are considered (e.g., the existence and uniqueness of the solution). We will supplement this area with the specific solutions we obtained.

The results obtained allow us to continue research in several directions in the future. Firstly, by implementing numerical modeling, it is possible to study the stability and dependence of the parameters of the obtained solution. Secondly, the transition to multidimensional cases and, thirdly, the consideration of nonlinear variants of the problem are important and promising areas. These areas allow us to expand the scope of solving applied problems of physics and engineering by combining analytical and numerical methods.

Conclusion

In this study, an analytical solution to a boundary value problem for a differential equation of mixed type with a fractional derivative of Caputo order was obtained. It was verified that the obtained solution satisfies the equation and the boundary conditions, and it was proven that this is a unique solution. It was also shown that in the limiting case of fractional order (when the order of the derivative is 1), this solution becomes a solution to the classical equation.

Theoretically, the conclusions obtained in the work complement the theory of fractional differential equations. It is shown that for equations of mixed type it is possible to solve a boundary value problem using fractional derivatives, and the classical results are generalized to fractional order. From an applied point of view, the obtained results can contribute to the accurate modeling of complex processes in various fields. For example, fractional-order models are currently used to describe phenomena with memory properties, such as relaxation or anomalous diffusion in viscoelastic media. The analytical solutions obtained in the course of the study make it possible to increase the accuracy of such models and analyze their behavior.

The obtained solution is expressed in analytical form. The main advantage of such a specific solution is that the behavior of the solution can be analyzed directly by freely changing the parameters of the problem (for example, the value of the order of the fractional derivative). In this case, the analytical solution serves as a standard for checking the correctness of the results obtained using numerical methods, allowing one to evaluate the accuracy of numerical models. One of the directions of future research is the extension of the proposed method to similar problems in multidimensional areas. The second direction is a comprehensive consideration of the nonlinear case of the equation. In addition, an important direction for the future is the development of numerical modeling methods for complex problems that cannot be solved analytically.

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