

O.S. Akhmetova^{1*} , S.A. Issayev² 

¹Almaty brunch of Saint-Peterburg University of the Humanities and Social Sciences,
Almaty, Kazakhstan

²Kazakh National Women's Teacher Training University, Almaty, Kazakhstan

*e-mail: ah_oksa@mail.ru

STATIONARY DIFFUSION MODEL IN HETEROGENEOUS MEDIA WITH SMALL PARAMETERS

Abstract

This study presents a theoretical and applied investigation of a stationary diffusion problem in heterogeneous media incorporating small parameters and modified boundary conditions. The relevance of this work stems from the necessity to accurately represent boundary layer effects in complex physical systems with variable internal structures, such as multilayered materials, biological tissues, and natural geophysical environments. Traditional models often oversimplify such effects, resulting in distortions in the solution near domain boundaries. The primary objective of the research is to construct a rigorous mathematical framework that accounts for both the presence of small parameters and altered boundary conditions. The analysis includes deriving a priori stability estimates and constructing asymptotic expansions for the solution. The methodology relies on functional formulation in Sobolev spaces, the Galerkin method, asymptotic techniques, and weak convergence theory. Special attention is given to the existence and uniqueness of both strong and generalized solutions, as well as to their convergence properties as the small parameters tend to zero. The results confirm the solvability of the problem, provide robust energy estimates, and establish solution stability under various types of boundary modifications. It is demonstrated that changes in boundary conditions primarily affect solution behavior near the domain edges, while preserving the global structure of the field. The model is applied in various fields, including climate modeling, thermal diffusion, and hydrophysical systems. This work lays a foundation for the development of adaptive numerical schemes and for extending the analysis to non-stationary problems.

Keywords: stationary diffusion, heterogeneous media, small parameters, boundary conditions, asymptotic analysis, solution stability, functional methods, a priori estimates

О.С. Ахметова¹, С.А. Исаев²

¹Санкт-Петербург Гуманитарлық кәсіподақтар университеті Алматы филиалы,
Алматы қ., Қазақстан

²Қазақ ұлттық қыздар педагогикалық университеті, Алматы қ., Қазақстан

ШАҒЫН ПАРАМЕТРЛЕРІ БАР БІРТЕКТІ ЕМЕС ОРТАДАҒЫ СТАЦИОНАРЛЫ ДИФФУЗИЯ МОДЕЛІ

Аңдатпа

Бұл зерттеу шекаралық шарттарды өзгерту шарты бойынша шағын параметрлері бар біртекті емес орталарда стационарлық диффузия мәселесін теориялық және қолданбалы талдауға арналған. Жұмыстың өзектілігі көп қабатты материалдар, биологиялық ұлпалар және табиғи геофизикалық түзілімдер сияқты органдың ауыспалы құрылымы бар физикалық жүйелерде туындайтын шекаралық әсерлерді дәл сипаттау қажеттілігімен байланысты. Классикалық модельдерде мұндай әсерлер жиі еленбейді немесе жеңілдетіледі, бұл шекаралық аймақтардағы ерітіндінің таралуының бұрмалануына әкеледі. Зерттеудің мақсаты - шағын параметрлерді және өзгертілген шекаралық шарттарды ескеретін қатаң математикалық модельді құру, сонымен қатар тұрақтылықтың априорлық бағасын алу және шешімдердің асимптотикалық кеңеюін құру. Әдістеме функционалдық кеңістіктерде мәселені тұжырымдауды, Галеркин әдісін қолдануды, асимптотикалық талдауды және әлсіз конвергенция теориясын қамтиды. Жалпыланған және күшті шешімдерді құруға, сондай-ақ параметрлер нөлге бейім болғандықтан олардың жақындасу шарттарын зерттеуге ерекше назар аударылады. Нәтижесінде

модельдің шешілетіндігі дәлелденді, энергиялық бағалаулар алынады, шекаралық модификациялардың әртүрлі түрлері үшін шешімдердің тұрақтылығы расталады. Шекаралық шарттарды өзгерту негізінен өрістің жалпы құрылымын бұзбай, шекараларға жақын шешімдердің эрекетіне әсер ететіні көрсетілген. Климатологияда, жылу алмасуды модельдеуде және гидрофизикада модельді қолдану мысалдары келтірілген. Жұмыс адаптивті сандық сұлбаларды әзірлеуге және талдауды стационарлық емес есептерге кеңейтуге негіз бола алады.

Түйін сөздер: стационарлы диффузия, біртекті емес орта, шағын параметрлер, шекаралық шарттар, асимптотикалық талдау, шешімдердің тұрақтылығы, функционалдық тәсіл, априорлық бағалау.

О.С. Ахметова¹, С.А. Исаев²

¹Алматинский филиал Санкт-Петербургского Гуманитарного университета профсоюзов,
г. Алматы, Казахстан

²Казахский национальный женский педагогический университет, г. Алматы, Казахстан

МОДЕЛЬ СТАЦИОНАРНОЙ ДИФФУЗИИ В НЕОДНОРОДНОЙ СРЕДЕ С МАЛЫМИ ПАРАМЕТРАМИ

Аннотация

Настоящее исследование посвящено теоретическому и прикладному анализу стационарной диффузионной задачи в неоднородных средах с малыми параметрами при условии модификации краевых условий. Актуальность работы обусловлена необходимостью точного описания граничных эффектов, возникающих в физических системах с переменной структурой среды, таких как многослойные материалы, биологические ткани и природные геофизические образования. В классических моделях такие эффекты зачастую игнорируются или упрощаются, что приводит к искажению распределения решения в пограничных зонах. Целью исследования является построение строгой математической модели, учитывающей малые параметры и изменённые граничные условия, а также получение априорных оценок устойчивости и построение асимптотических разложений решений. Методология включает в себя постановку задачи в функциональных пространствах, применение метода Галеркина, асимптотического анализа и теории слабой сходимости. Особое внимание уделено построению обобщённых и сильных решений, а также исследованию условий их сходимости при стремлении параметров к нулю. В результате доказана разрешимость модели, получены энергетические оценки и подтверждена устойчивость решений при различных видах граничных модификаций. Показано, что модификация краевых условий влияет преимущественно на поведение решений вблизи границ, не нарушая общей структуры поля. Представлены примеры применения модели в задачах климатологии, моделировании теплопереноса и гидрофизике. Работа может служить основой для разработки адаптивных численных схем и расширения анализа на нестационарные задачи.

Ключевые слова: стационарная диффузия, неоднородная среда, малые параметры, краевые условия, асимптотический анализ, устойчивость решений, функциональный подход, априорные оценки.

Main Provisions

This paper investigates the influence of boundary condition modification on the behavior of solutions of a stationary diffusion problem in inhomogeneous media with small parameters. A theorem of the existence of strong and generalized solutions of the problem is established, a priori stability estimates are obtained, and asymptotic expansions with control of the accuracy of approximations are constructed. It is revealed that modifications of boundary conditions significantly affect the distribution of the solution in the boundary region. The methodology is substantiated using the theory of boundary value problems and methods of functional analysis.

Introduction

Mathematical modeling of stationary processes of matter and heat transfer in environments with spatial heterogeneity and small parameters is an important area of applied analysis. Similar problems arise in a wide range of scientific and engineering applications - from thermal physics and geophysics to biomechanics and materials science. A characteristic feature of such processes is the increased

sensitivity of solutions to boundary conditions, especially in situations where the properties of the medium change significantly near the boundary of the region.

Traditional formulations of diffusion problems often assume strictly specified boundary conditions and constant coefficients, which may not be justified in real systems. Classical approaches developed in the works of O.A. Ladyzhenskaya [1, p. 15] provide a fundamental basis, but do not cover the influence of structural heterogeneities and variations at the boundaries. More modern studies suggest using asymptotic analysis and homogenization methods to account for the heterogeneous microstructure [2, p. 47; 3, p. 92]. However, the issue of modifying boundary conditions in the presence of small parameters remains insufficiently studied. In climate modeling problems, for example, when assessing heat transfer through ice cover, it is essential to consider the transition zones between layers with different thermal conductivities, such as those between air and ice, and ice and water. Here, both spatial heterogeneity and weak heat exchange at the boundaries become unavoidable, which cannot be correctly described without introducing modified boundary conditions [4, p. 213]. In this context, the development of models that combine a functional-analytical approach with asymptotic stability of solutions becomes relevant.

The present study aims to construct a mathematical model of stationary diffusion that takes into account small parameters, structural heterogeneity of the medium, and modified boundary conditions. A rigorous justification for the existence and stability of solutions in functional spaces is proposed, a priori estimates are obtained, and asymptotic expansions are constructed. In addition, the practical application of the model for describing a stable temperature profile in ice layers with various types of boundary interactions is demonstrated. Thus, the developed model has both theoretical value and practical significance for heat exchange problems in natural and engineering systems.

Research methodology

The study is theoretical and analytical, based on modern methods in the theory of boundary value problems for elliptic equations. The mathematical model is formulated based on a stationary diffusion problem, where the heterogeneity of the medium and the presence of small parameters determine spatial variations in the coefficients and modifications to the boundary conditions. The central place is occupied by a strict formulation of the problem in Sobolev spaces, which allows analyzing solutions within the framework of both intense and generalized interpretations. To substantiate the existence and uniqueness of solutions, Galerkin and variational calculus methods were used, as well as the weak convergence technique, which made it possible to prove the solvability of the problem rigorously. All conclusions are accompanied by a priori energy estimates based on classical functional inequalities - Cauchy, Young, Sobolev - and the maximum principle. This approach ensured the mathematical rigor and stability of the model. Particular attention in the work is paid to the construction of asymptotic expansions of solutions in powers of a small parameter. By formally substituting these expansions into the original equations and modified boundary conditions, equations for each order of approximation were identified, which made it possible to determine the exact structure of the correction terms. The obtained estimates of the residual terms confirmed the convergence of the approximations and the stability of the solution as $\varepsilon \rightarrow 0$. An important step was the justification of the correctness of the limit transition as small parameters tend to zero. The use of weak convergence concepts in Sobolev spaces ensured a rigorous analysis of the behavior of solutions, including nonlinear boundary terms. Thus, the proposed methodological approach combines functional and analytical rigor with the ability to construct stable approximations, providing a comprehensive understanding of the influence of modifying boundary conditions on the structure of a stationary solution in an inhomogeneous medium.

Statement of the Problem

A boundary value problem is considered that models the steady motion of a viscous incompressible fluid with variable density in a limited region $\Omega \in R^3$. The motion is described by the following system of equations:

$$\rho[(\vec{v} \cdot \nabla)\vec{v}] = \nu\Delta\vec{v} - \nabla p + \lambda(\nabla\rho \cdot \nabla)\vec{v} + \rho\vec{f}, \quad (1)$$

This is the generalized equation of motion (modified Navier–Stokes equation) for an incompressible fluid with variable density.

The left-hand side is the inertial term: the density ρ times the convective derivative of the velocity $(\vec{v} \cdot \nabla)\vec{v}$.

The left-hand side is the inertial term: the density ρ multiplied by the convective derivative of the velocity $(\vec{v} \cdot \nabla)$.

The right side includes:

$(\vec{v} \cdot \nabla)$ – a viscous term describing internal friction (diffusion of momentum);

∇p – a pressure gradient;

$\lambda(\nabla\rho \cdot \nabla)\vec{v}$ – an additional term reflecting the effect of the density gradient on the velocity;

$\rho\vec{f}$ – a mass force (for example, gravity) multiplied by the density.

Condition of incompressibility of the medium (2): the divergence of the velocity vector field is zero. This means that the volume of the liquid is preserved.

$$\operatorname{div}\vec{v} = 0 \quad (2)$$

The equation of density (3) transfer takes into account diffusion. The left part is the convective transfer of density by the velocity field, and the right part is the diffusion spreading of density with the coefficient λ . The equation describes the stationary distribution of density in a moving medium.

$$(\vec{v} \cdot \nabla)\rho = \lambda\Delta\rho \quad (3)$$

Boundary conditions:

$$\vec{v}|_{\partial\Omega} = 0, \quad \rho|_{\partial\Omega} = \rho_0(x), \quad \int_{\Omega} p \, dx = 0 \quad (4)$$

where

$\vec{v}|_{\partial\Omega} = 0$ –no-slip condition: the liquid is at rest at the boundary of the region;

$\rho|_{\partial\Omega} = \rho_0(x)$ –the density distribution at the boundary of the region is given;

$\int_{\Omega} p \, dx = 0$ –pressure normalization: the average pressure value over the region is assumed to be zero (for uniqueness of the solution).

Description of variables:

\vec{v} –volumetric flow velocity vector;

p – scalar pressure field;

ρ –scalar function of the density of a liquid (or solution);

λ –molecular density diffusion coefficient;

\vec{f} – vector of mass force acting in a region (for example, gravity);

ν – kinematic viscosity coefficient.

The function $\rho_0(x)$ defined on the boundary $\partial\Omega$ satisfies the condition

$$|\rho_0(x)| \leq M \quad (5)$$

Definition 1. A strong solution to problem (1)-(4) is a function $\vec{v}(x) \in W_2^2(\Omega) \cap (W_2^1(\Omega), \rho(x) \in W_2^2(\Omega), \operatorname{div} \vec{v} = 0, p(x) \in W_2^1(\Omega)$, satisfying (1)-(4) to the appropriate extent.

Theorem 1. Let $\partial\Omega \subset C^2$, $\vec{f}(x) \in L_2(\Omega)$, $\rho_0(x) \in W_2^{3/2}(\Omega)$ and condition (5) be satisfied. Then there exists at least one strong solution to problem (1)-(4).

Proof. We obtain a priori estimates of the solution. By the maximum principle for elliptic equations, we have

$$|\rho| \leq M \quad (6)$$

Let us multiply (1) by \vec{v} in $L_2(\Omega)$. Considering that by virtue of (3)

$$\int_{\Omega} [\rho(\vec{v} \cdot \nabla)\vec{v} - \lambda(\nabla\rho \cdot \nabla)\vec{v}] \vec{v} d\Omega = 0,$$

we get the inequality

$$v \|\nabla\vec{v}\|_{L_2(\Omega)}^2 \leq CM \|\vec{f}\|_{W_2^{-1}(\Omega)} \|\nabla\vec{v}\|_{L_2(\Omega)},$$

from which follows the assessment

$$v \|\nabla\vec{v}\|_{L_2(\Omega)}^2 \leq C \|\vec{f}\|_{W_2^{-1}(\Omega)} \quad (7)$$

Now, multiplying (3) by $\rho - \rho_0$ in $L_2(\Omega)$, we get

$$\lambda \|\nabla\rho\|_{L_2(\Omega)}^2 \leq C \left(\|\vec{v}\|_{L_2(\Omega)} \|\nabla\rho\|_{L_2(\Omega)} \max_{x \in \Omega} |\rho - \rho_0| + \lambda \|\nabla\rho\|_{L_2(\Omega)} \|\nabla\rho_0\|_{L_2(\Omega)} \right)$$

Applying Young's inequality on the right-hand side

$$\begin{aligned} & \|\vec{v}\|_{L_2(\Omega)} \|\nabla\rho\|_{L_2(\Omega)} \max_{x \in \Omega} |\rho - \rho_0| + \lambda \|\nabla\rho\|_{L_2(\Omega)} \|\nabla\rho_0\|_{L_2(\Omega)} \leq \\ & \leq \frac{\delta_1}{2} \|\nabla\rho\|_{L_2(\Omega)}^2 + \frac{\delta_1^{-1}}{2} \left[\|\vec{v}\|_{L_2(\Omega)}^2 \max_{x \in \Omega} |\rho - \rho_0|^2 + \lambda^2 \|\nabla\rho_0\|_{L_2(\Omega)}^2 \right], \end{aligned}$$

For sufficiently small δ_1 we have

$$\lambda \|\nabla\rho\|_{L_2(\Omega)}^2 \leq C \left(\|\vec{f}\|_{W_2^{-1}(\Omega)} + \|\nabla\rho_0\|_{L_2(\Omega)}^2 \right) \quad (8)$$

Next, using the theory of boundary value problems for elliptic equations, from equation (3) we obtain

$$\|\nabla\rho\|_{L_2(\Omega)} \leq C \left(\|(\vec{v} \cdot \nabla)\rho\|_{L_2(\Omega)} + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)} \right).$$

Using inequality

$$\|\nabla\rho\|_{L_4(\Omega)} \leq C \|\rho\|_{W_2^2(\Omega)}^{1/2} \|\rho\|_{L_\infty(\Omega)}^{1/2} \leq C \|\rho\|_{W_2^2(\Omega)}^{1/2} M^{1/2}$$

Let's estimate the first term on the right-hand side

$$\begin{aligned} \|(\vec{v} \cdot \nabla)\rho\|_{L_2(\Omega)} & \leq \|\vec{v}\|_{L_4(\Omega)} \|\nabla\rho\|_{L_4(\Omega)} \leq C \|\vec{v} \cdot \nabla\|_{L_2(\Omega)} M^{1/2} \|\rho\|_{W_2^2(\Omega)}^{1/2} \leq \\ & \leq \frac{\delta_2}{2} \|\rho\|_{W_2^2(\Omega)} + \frac{\delta_2^{-1}}{2} C \|\vec{v} \cdot \nabla\|_{L_2(\Omega)}^2 M. \end{aligned}$$

For sufficiently small δ_2 , we establish

$$\|\rho\|_{W_2^2(\Omega)} \leq C \left(\|\vec{f}\|_{W_2^{-1}(\Omega)} + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)} \right) \quad (9)$$

Let us rewrite (1) as

$$v\Delta\vec{v} - \nabla p = \rho(\vec{v} \cdot \nabla)\vec{v} - \lambda(\nabla\rho \cdot \nabla)\vec{v} - \rho\vec{f},$$

Let's estimate the right side as follows:

$$\begin{aligned} & \|\rho(\vec{v} \cdot \nabla)\vec{v} - \lambda(\nabla\rho \cdot \nabla)\vec{v} - \rho\vec{f}\|_{L_2(\Omega)} \leq M\|\vec{v}\|_{L_4(\Omega)}\|\nabla\vec{v}\|_{L_4(\Omega)} + \\ & \lambda\|\nabla\rho\|_{L_4(\Omega)}\|\nabla\vec{v}\|_{L_4(\Omega)} + M\|\vec{f}\|_{L_2(\Omega)} \leq \\ & \leq C(M\|\nabla\vec{v}\|_{L_2(\Omega)}^{5/4}\|\vec{v}\|_{W_2^2(\Omega)}^{3/4} + \lambda\|\rho\|_{W_2^2(\Omega)}^{1/2}M^{1/2}\|\vec{v}\|_{W_2^2(\Omega)}^{3/4}\|\nabla\vec{v}\|_{L_2(\Omega)}^{1/4} + M\|\vec{f}\|_{L_2(\Omega)}) \leq \\ & \leq \frac{3}{4}\delta_3\|\vec{v}\|_{W_2^2(\Omega)} + \frac{1}{4}\delta_3^{-3}C\left(M^4\|\nabla\vec{v}\|_{L_2(\Omega)}^5 + \lambda^4\|\rho\|_{W_2^2(\Omega)}^2M^2\|\nabla\vec{v}\|_{L_2(\Omega)}\right) + CM\|\vec{f}\|_{L_2(\Omega)} \end{aligned}$$

Using estimates of solutions to boundary value problems for the Navier-Stokes system, we find

$$\|\vec{v}\|_{W_2^2(\Omega)} + \|\nabla\rho\|_{L_4(\Omega)} \leq C\left(\|\vec{f}\|_{L_2(\Omega)} + \|\vec{f}\|_{W_2^{-1}(\Omega)}^5 + \|\vec{f}\|_{W_2^{-1}(\Omega)} + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)}^2\right) \quad (10)$$

The solvability of problem (1)-(4) can be proved based on estimates (6)-(10), for example, by the Galerkin method. The proof is complete.

Let us consider various approximations of problem (1)-(4). Let us start with the system

$$\rho^\varepsilon(\vec{v}^\varepsilon \cdot \nabla)\vec{v}^\varepsilon = \nu\Delta\vec{v}^\varepsilon + \frac{1}{\varepsilon}\nabla\operatorname{div}\vec{v}^\varepsilon - \frac{1}{2}\rho^\varepsilon\vec{v}^\varepsilon\operatorname{div}\vec{v}^\varepsilon + \rho^\varepsilon\vec{f} + \lambda(\nabla\rho^\varepsilon \cdot \nabla)\vec{v}^\varepsilon, \quad (11)$$

$$(\vec{v}^\varepsilon \cdot \nabla)\rho^\varepsilon = \lambda\Delta\rho^\varepsilon, \quad (12)$$

with boundary conditions

$$\vec{v}^\varepsilon|_{\partial\Omega} = 0, \quad \rho^\varepsilon|_{\partial\Omega} = \rho_0(x). \quad (13)$$

Theorem 2. Let $\vec{f}(x) \in L_2(\Omega)$, $\rho_0(x) \in W_2^{3/2}(\Omega)$ and satisfy (5), $\partial\Omega \in C^2$. Then there exists at least one solution to problem (11) – (13) $\vec{v}^\varepsilon(x) \in W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$, $\rho^\varepsilon(x) \in W_2^2(\Omega)$ the estimate holds

$$\|\vec{v}^\varepsilon\|_{W_2^2(\Omega)} + \frac{1}{\varepsilon}\|\nabla\operatorname{div}\vec{v}^\varepsilon\|_{L_2(\Omega)} + \|\rho^\varepsilon\|_{W_2^2(\Omega)} \leq C_1\left(\|\vec{f}\|_{L_2(\Omega)}\|\rho_0\|_{W_2^{3/2}(\Omega)}\right) \quad (14)$$

Proof. As before, $|\rho^\varepsilon| \leq M$.

Multiply (11) by \vec{v}^ε in $L_2(\Omega)$. By the equality

$$\int_{\Omega} \left[\rho^\varepsilon(\vec{v}^\varepsilon \cdot \nabla)\vec{v}^\varepsilon + \frac{1}{2}\rho^\varepsilon\vec{v}^\varepsilon\operatorname{div}\vec{v}^\varepsilon - \lambda(\nabla\rho^\varepsilon \cdot \nabla)\vec{v}^\varepsilon \right] \vec{v}^\varepsilon dx = 0,$$

we will receive

$$\nu\|\nabla\vec{v}^\varepsilon\|_{L_2(\Omega)}^2 + \frac{1}{\varepsilon}\|\nabla\operatorname{div}\vec{v}^\varepsilon\|_{L_2(\Omega)}^2 \leq C\|\vec{f}\|_{W_2^{-1}(\Omega)}^2 \quad (15)$$

Next, from equation (12) we have

$$\|\rho\|_{W_2^2(\Omega)} \leq C_1\left(\|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\Omega)}\right) \quad (16)$$

Let us rewrite (11) as

$$\nu \nabla \vec{v}^\varepsilon + \frac{1}{\varepsilon} \nabla \operatorname{div} \vec{v}^\varepsilon = \rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon + \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon - \lambda (\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon - \rho^\varepsilon \vec{f},$$

Estimating the right-hand side in $L_2(\Omega)$ taking into account the inequality of the ε -approximation of the Navier-Stokes equations

$$\|\vec{v}^\varepsilon\|_{W_2^2(\Omega)} + \frac{1}{\varepsilon} \|\nabla \operatorname{div} \vec{v}^\varepsilon\|_{L_2(\Omega)} \leq C \left\| \nu \nabla \vec{v}^\varepsilon + \frac{1}{\varepsilon} \nabla \operatorname{div} \vec{v}^\varepsilon \right\|_{L_2(\Omega)},$$

we will receive

$$\|\vec{v}^\varepsilon\|_{W_2^2(\Omega)} + \frac{1}{\varepsilon} \|\nabla \operatorname{div} \vec{v}^\varepsilon\|_{L_2(\Omega)} \leq C \left(\|\vec{f}\|_{L_2(\Omega)} + \|\vec{f}\|_{W_2^{-1}(\Omega)}^5 + \|\vec{f}\|_{W_2^{-1}(\Omega)} \|\rho_0\|_{W_2^{3/2}(\partial\Omega)}^2 \right) \quad (17)$$

Estimates (16), (17) guarantee the validity of Theorem 2. In addition, it follows from them that it is possible to select subsequences $\{\rho^\varepsilon\}, \{\vec{v}^\varepsilon\}, \left\{\frac{1}{\varepsilon} \operatorname{div} \vec{v}^\varepsilon\right\}$ that weakly converge in the corresponding measure as $\varepsilon \rightarrow 0$. The proof is complete.

Let us rewrite (11) in the form

$$\begin{aligned} \rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon &= \nu \Delta \vec{v}^\varepsilon - \nabla p^\varepsilon - \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon + \rho^\varepsilon \vec{f} + \lambda (\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon, \\ \operatorname{div} \vec{v}^\varepsilon + \varepsilon \rho^\varepsilon &= 0. \end{aligned} \quad (18)$$

Theorem 3. Let the conditions of Theorem 2 be satisfied. In addition to the fact that $\vec{f}(x) \in W_2^{-1}(\Omega)$ и $\|\vec{f}\|_{L_2(\Omega)} \cdot M, \|\rho_0(x)\|_{W_2^{3/2}(\Omega)}$ are sufficiently small. Then the solution to problem (18), (12), (13) can be represented in the form

$$\begin{aligned} \vec{v}^\varepsilon &= \vec{v} + \sum_{k=1}^N \varepsilon^k \vec{v}_k + \bar{w}_\varepsilon \varepsilon^{N+1}, \\ p^\varepsilon &= p + \sum_{k=1}^N \varepsilon^k p_k + p_\varepsilon \varepsilon^{N+1}, \\ \rho^\varepsilon &= \rho + \sum_{k=1}^N \varepsilon^k \rho_k + \rho_\varepsilon \varepsilon^{N+1}, \end{aligned} \quad (19)$$

where \vec{v}, p, ρ is the solution to problem (1) – (4), and the following estimates are valid

$$\begin{aligned} \|\vec{v}_k\|_{W_2^2(\Omega)} + \|\nabla p_k\|_{L_2(\Omega)} + \|\rho_k\|_{W_2^2(\Omega)} &\leq C_2, \\ \|\bar{w}_\varepsilon\|_{W_2^2(\Omega)} + \|\nabla p_\varepsilon\|_{L_2(\Omega)} + \|\rho_\varepsilon\|_{W_2^2(\Omega)} &\leq C_3. \end{aligned} \quad (20)$$

Proof. Substitute the expansion (19) into (18), (12). Collecting the coefficients at the powers of ε , for all $k = 1, 2, \dots, N$ we obtain

$$\begin{aligned} \rho_0 (\vec{v}_0 \cdot \nabla) \vec{v}_k + \rho_0 (\vec{v}_k \cdot \nabla) \vec{v}_0 + \rho_k (\vec{v}_0 \cdot \nabla) \vec{v}_0 + \frac{1}{2} \rho_0 \vec{v}_0 \operatorname{div} \vec{v}_k &= \\ = \nu \Delta \vec{v}_k - \nabla p_k + \rho_k \vec{f} + \lambda [(\nabla \rho_0 \cdot \nabla) \vec{v}_k + (\nabla \rho_k \cdot \nabla) \vec{v}_0] - \vec{h}_k(x), \end{aligned} \quad (21)$$

$$h_k(x) = \sum_{\substack{i,j,\ell=1 \\ i+j+\ell=k-1}}^{k-1} \left[\rho_i(\vec{v}_j \cdot \nabla) \vec{v}_\ell + \frac{1}{2} \rho_i \vec{v}_j \operatorname{div} \vec{v}_i \right] + \sum_{\substack{i,j,\ell=1 \\ i+j+\ell=k-1}}^{k-1} \lambda(\nabla \rho_i \cdot \nabla) \vec{v}_j,$$

$$\operatorname{div} \vec{v}_k + p_{k+1} = 0, \quad (22)$$

$$\rho_k(\vec{v}_0 \cdot \nabla) + \rho_0(\vec{v}_k \cdot \nabla) + g_k(x) = \lambda \Delta \rho_k, \quad (23)$$

$$g_k(x) = \sum_{\substack{i,j,\ell=1 \\ i+j+\ell=k-1}}^{k-1} (\vec{v}_j \cdot \nabla) \rho_j,$$

$$\vec{v}_k|_{\partial\Omega} = 0, \quad \rho_k|_{\partial\Omega} = 0. \quad (24)$$

The functions $\vec{w}_\varepsilon, p_\varepsilon, \rho_\varepsilon$ are the solution to the problem

$$\rho_0(\vec{v}_0 \cdot \nabla) \vec{w}_\varepsilon + \rho_0(\vec{w}_\varepsilon \cdot \nabla) \vec{v}_0 + \rho_\varepsilon(\vec{v}_0 \cdot \nabla) \vec{v}_0 + \frac{1}{2} \rho_0 \vec{v}_0 \operatorname{div} \vec{w}_\varepsilon =$$

$$= \nu \Delta \vec{w}_\varepsilon - \nabla p_\varepsilon + \rho_\varepsilon \vec{f} + \lambda[(\nabla \rho_0 \cdot \nabla) \vec{w}_\varepsilon + (\nabla \rho_\varepsilon \cdot \nabla) \vec{v}_0] - \vec{h}_{N+1}(x), \quad (25)$$

$$h_{N+1}(x) = \sum_{\substack{i,j,\ell=1 \\ i+j+\ell>N}}^{k-1} \varepsilon^{i+j+\ell-N} \left[\rho_i(\vec{v}_j \cdot \nabla) \vec{v}_\ell + \frac{1}{2} \rho_i \vec{v}_j \operatorname{div} \vec{v}_i \right] + \sum_{\substack{i,j,\ell=1 \\ i+j+\ell=k-1}}^{k-1} \varepsilon^{i+j} \lambda(\nabla \rho_i \cdot \nabla) \vec{v}_j,$$

$$\operatorname{div} \vec{w}_\varepsilon + \varepsilon p_\varepsilon + \varepsilon p_N = 0, \quad (26)$$

$$\rho_\varepsilon(\vec{v}_0 \cdot \nabla) + \rho_0(\vec{w}_\varepsilon \cdot \nabla) + g_{N+1}(x) = \lambda \Delta \rho_k, \quad (27)$$

$$g_{N+1}(x) = \sum_{\substack{i,j,\ell=1 \\ i+j+\ell>N}}^N \varepsilon^{i+j} (\vec{v}_i \cdot \nabla) \rho_j,$$

$$\vec{w}_\varepsilon|_{\partial\Omega} = 0, \quad \rho_\varepsilon|_{\partial\Omega} = 0. \quad (28)$$

Problems (21) – (24) for each fixed $k = 1, 2, \dots, N$, (25) – (28) have solutions

$$\vec{v}_k(x), \vec{w}_\varepsilon(x) \in W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$$

$$\nabla \rho_k(x), \nabla \rho_\varepsilon(x) \in L_2(\Omega), \rho_k(x), \rho_\varepsilon(x) \in W_2^2(\Omega),$$

where \vec{v}_k, ρ_k, p_k do not depend on ε .

Let us estimate the functions $g_k(x), \vec{h}_k(x)$

$$\|g_k\|_{L_2(\Omega)} \leq \sum_{\substack{i,j=1 \\ i+j=k-1}}^{k-1} \|\vec{v}_i\|_{L_2(\Omega)} \|\nabla \rho_j\|_{L_2(\Omega)} \leq$$

$$(29)$$

$$\begin{aligned}
 &\leq C \sum_{i+j=k-1}^{k-1} \|\nabla \vec{v}_i\|_{L_2(\Omega)} \|\Delta \rho_j\|_{L_2(\Omega)}^{3/4} \|\rho_j\|_{L_2(\Omega)}^{1/4} \\
 \|\vec{h}_k\|_{L_2(\Omega)} &\leq \sum_{i,j=1}^{k-1} \left(M \|\vec{v}_i\|_{L_6(\Omega)} \|\nabla \vec{v}_j\|_{L_3(\Omega)} + \lambda \|\nabla \rho_i\|_{L_4(\Omega)} \|\nabla \vec{v}_j\|_{L_4(\Omega)} \right) \leq \\
 &\leq \sum_{i,j=1}^{k-1} \left(M \|\nabla \vec{v}_i\|_{L_2(\Omega)} \|\Delta \vec{v}_j\|_{L_2(\Omega)}^{1/2} \|\nabla \vec{v}_j\|_{L_2(\Omega)}^{1/2} + \lambda \|\nabla \rho_i\|_{L_2(\Omega)}^{7/8} \|\rho_i\|_{L_2(\Omega)}^{1/8} \|\Delta \vec{v}_j\|_{L_2(\Omega)}^{7/8} \|\vec{v}_j\|_{L_2(\Omega)}^{1/8} \right)
 \end{aligned} \tag{30}$$

From equations (21) – (23) we have

$$\begin{aligned}
 v \|\Delta \vec{v}_k\|_{L_2(\Omega)} + \|\nabla p_k\|_{L_2(\Omega)} &\leq C [\|\nabla p_{k-1}\| + M (|\vec{v}_0|_\Omega \|\nabla \vec{v}_k\|_{L_2(\Omega)} + \|\vec{v}_k\|_{L_2(\Omega)} |\nabla \vec{v}_0|_\Omega) + \\
 + \|\rho_k\|_{L_2(\Omega)} |\vec{v}_0|_\Omega |\nabla \vec{v}_0|_\Omega + \|\rho_k\|_{L_3(\Omega)} \|\vec{f}\|_{L_6(\Omega)} + \lambda (\|\nabla \rho_0\|_{L_6(\Omega)} |\nabla \vec{v}_k|_{L_3(\Omega)} + \|\nabla \rho_k\|_{L_2(\Omega)} |\nabla \vec{v}_0|_\Omega) + \\
 + \|\vec{h}_k\|_{L_2(\Omega)}] &\leq \delta (\|\Delta \rho_k\|_{L_2(\Omega)} + \|\Delta \vec{v}_k\|_{L_2(\Omega)}) + \\
 + C [\|\nabla p_{k-1}\|_{L_2} + \|\vec{h}_k\|_{L_2(\Omega)} + (1 + \delta^{-1}) (\|\rho_k\|_{L_2(\Omega)} + \|\vec{v}_k\|_{L_2(\Omega)}) \times \\
 \times (1 + |\vec{v}_0|_\Omega^2 + |\nabla \vec{v}_0|_\Omega^2 + \|\vec{f}\|_{L_6(\Omega)}^2 + \|\nabla \rho_0\|_{L_6(\Omega)}^4)] &
 \end{aligned} \tag{31}$$

$$\begin{aligned}
 \|\Delta \rho_k\|_{L_2(\Omega)} &\leq C [|\vec{v}_0|_\Omega \|\nabla \rho_k\|_{L_2(\Omega)} + \|\vec{v}_k\|_{L_3(\Omega)} \|\nabla \rho_0\|_{L_6(\Omega)} + \|\vec{g}_k\|_{L_2(\Omega)}] \leq \\
 &\leq \delta (\|\nabla \rho_k\|_{L_2(\Omega)} + \|\Delta \vec{v}_k\|_{L_2(\Omega)}) + \\
 + C_\delta [|\vec{v}_0|_\Omega^4 \|\rho_k\|_{L_2(\Omega)} + \|\vec{v}_k\|_{L_2(\Omega)} \|\nabla \rho_0\|_{L_6(\Omega)}^{4/3} + \|\vec{g}_k\|_{L_2(\Omega)}] &
 \end{aligned} \tag{32}$$

The function $\vec{h}_{N+1}(x)$, $g_{N+1}(x)$, $\vec{w}_\varepsilon(x)$, $p_\varepsilon(x)$, $\rho_\varepsilon(x)$ are estimated similarly. From inequalities of the form (29) – (32) follows estimate (20) for sufficiently small $\vec{f}(x)$ and $\rho_0(x)$. The proof is complete.

Let us consider the following approximation of equations (1) – (4)

$$\rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon = v \Delta \vec{v}^\varepsilon - \nabla p^\varepsilon + \lambda (\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon + \rho^\varepsilon \vec{f} - \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon, \tag{33}$$

$$\operatorname{div} \vec{v}^\varepsilon = \varepsilon (\Delta p^\varepsilon - p^\varepsilon), \tag{34}$$

$$(\vec{v}^\varepsilon \cdot \nabla) \rho^\varepsilon = \lambda \Delta \rho^\varepsilon, \tag{35}$$

with boundary conditions

$$\vec{v}^\varepsilon|_{\partial\Omega} = 0, \quad \frac{\partial p^\varepsilon}{\partial n} \Big|_{\partial\Omega} = 0, \quad \rho^\varepsilon|_{\partial\Omega} = \rho_0(x). \tag{36}$$

Definition 2. A generalized solution to problem (33) – (36) is a function $\vec{v}^\varepsilon(x) \in \dot{W}_2^1(\Omega)$, $\rho^\varepsilon(x) \in W_2^2(\Omega)$ that satisfies the integral identities

$$\int_{\Omega} \left[\rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon \vec{\psi} + \nu \nabla \vec{v}^\varepsilon \nabla \vec{\psi} - p^\varepsilon \operatorname{div} \vec{\psi} - \lambda (\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon \vec{\psi} + \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon \vec{\psi} - \rho^\varepsilon \vec{f} \vec{\psi} \right] dx = 0,$$

$$\int_{\Omega} [\operatorname{div} \vec{v}^\varepsilon \varphi + \varepsilon (\nabla p^\varepsilon \nabla \varphi + p^\varepsilon \varphi)] dx = 0,$$

for any $\vec{\psi} \in W_2^1(\Omega)$, $\varphi(x) \in \dot{W}_2^1(\Omega)$ and the problem $(\vec{v}^\varepsilon \cdot \nabla) \rho^\varepsilon = \lambda \Delta \rho^\varepsilon$, $\rho^\varepsilon|_{\partial\Omega} = \rho_0(x)$ almost everywhere in $\bar{\Omega}$.

Definition 3. A strong solution of problem (33) – (36) is a function $\vec{v}^\varepsilon(x) \in W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$, $p^\varepsilon(x) \in W_2^2(\Omega)$, $\rho^\varepsilon(x) \in W_2^2(\Omega)$ satisfying (33) – (36) almost everywhere in $\bar{\Omega}$.

Theorem 4. $\vec{f}(x) \in W_2^{-1}(\Omega)$, $\rho_0(x) \in W_2^{3/2}(\partial\Omega)$, $\partial\Omega \in C^2$. Then problem (33) – (36) has at least one generalized solution and the estimate holds

$$\|\vec{v}^\varepsilon\|_{\dot{W}_2^1(\Omega)} + \|p^\varepsilon\|_{L_2(\Omega)} + \|\rho^\varepsilon\|_{W_2^2(\Omega)} \leq C \left(\|\vec{f}(x)\|_{W_2^{-1}(\Omega)} + \|\rho_0(x)\|_{W_2^{3/2}(\partial\Omega)} + 1 \right), \quad (37)$$

in which the constant C does not depend on ε .

Proof. Let us introduce estimate (37). Note that by virtue of the maximum principle for elliptic equations

$$|\rho^\varepsilon|_\Omega = \max_{x \in \bar{\Omega}} |\rho^\varepsilon| \leq \max_{x \in \bar{\Omega}} |\rho_0| \leq M.$$

Multiplying (33) by \vec{v}^ε in $L_2(\Omega)$, we obtain

$$\nu \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)}^2 - \int_{\Omega} p^\varepsilon \operatorname{div} \vec{v}^\varepsilon dx = \int_{\Omega} \left[\lambda (\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon + \rho^\varepsilon \vec{f} - \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon - \rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon \right] \vec{v}^\varepsilon dx$$

Using (34), (35), we find

$$\nu \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon \|\nabla p^\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon \|p^\varepsilon\|_{L_2(\Omega)}^2 \leq M \|\vec{f}\|_{W_2^{-1}(\Omega)} \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)}.$$

Using Cauchy's inequality, we obtain

$$\nu \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon \|\nabla p^\varepsilon\|_{L_2(\Omega)}^2 + \varepsilon \|p^\varepsilon\|_{L_2(\Omega)}^2 \leq C \|\vec{f}\|_{W_2^{-1}(\Omega)}^2$$

Multiplying (35) by $\rho^\varepsilon - \rho_0$ in $L_2(\Omega)$, we have

$$\lambda \|\nabla \rho^\varepsilon\|_{L_2(\Omega)}^2 = \frac{1}{2} \int_{\Omega} (\vec{v}^\varepsilon \cdot \nabla) (\rho^\varepsilon)^2 dx + \int_{\Omega} \nabla \rho^\varepsilon \cdot \nabla \rho_0 dx - \int_{\Omega} (\vec{v}^\varepsilon \cdot \nabla) \rho^\varepsilon \rho_0 dx$$

From this follows the inequality

$$\lambda \|\nabla \rho^\varepsilon\|_{L_2(\Omega)}^2 \leq C \left(\|\vec{f}\|_{W_2^{-1}(\Omega)}^2 \cdot M^2 + \|\nabla \rho_0\|_{L_2(\Omega)}^2 \right)$$

From the theory of boundary value problems for elliptic equations follows the estimate

$$\|\rho^\varepsilon\|_{W_2^2(\Omega)} \leq C \left(\|\rho_0(x)\|_{W_2^{3/2}(\partial\Omega)} + \|(\vec{v}^\varepsilon \cdot \nabla) \rho^\varepsilon\|_{L_2(\Omega)} \right) \leq$$

$$\leq C \left(\|\rho_0(x)\|_{W_2^{3/2}(\partial\Omega)} + \|\vec{v}^\varepsilon\|_{L_4(\Omega)} \|\rho^\varepsilon\|_{L_4(\Omega)} \right) \leq \frac{\delta_4}{2} \|\rho^\varepsilon\|_{W_2^2(\Omega)} + \\ + C \frac{\delta_4^{-1}}{2} \cdot M \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)}^2 + \|\rho_0(x)\|_{W_2^{3/2}(\partial\Omega)},$$

from here, for a sufficiently small δ_4 , we conclude

$$\|\rho^\varepsilon\|_{W_2^2(\Omega)} \leq C \left(\|\rho_0(x)\|_{W_2^{3/2}(\partial\Omega)} + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 \right).$$

Let us obtain an estimate for the function $p(x)$. To do this, we multiply (33) by the function $\vec{\varphi}(x) \in \dot{W}_2^1(\Omega)$. We have

$$\int_{\Omega} \nabla \rho^\varepsilon \vec{\varphi}(x) dx = \int_{\Omega} \left[\nu \Delta \vec{v}^\varepsilon + \lambda (\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon + \rho^\varepsilon \vec{f} - \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon - \rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon \right] \vec{\varphi}(x) dx$$

Using inequalities

$$\int_{\Omega} \nu \Delta \vec{v}^\varepsilon \vec{\varphi}(x) dx \leq C \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\vec{\varphi}\|_{W_2^1(\Omega)}$$

$$\int_{\Omega} \lambda (\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon \vec{\varphi}(x) dx \leq C \|\nabla \vec{v}^\varepsilon\|_{L_4(\Omega)} \|\nabla \rho^\varepsilon\|_{L_4(\Omega)} \|\vec{\varphi}\|_{L_4(\Omega)} \leq$$

$$\leq C \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\rho^\varepsilon\|_{W_2^{3/2}(\partial\Omega)}^{1/2} M^{1/2} \|\vec{\varphi}\|_{W_2^1(\Omega)},$$

$$\int_{\Omega} \left(\frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon - \rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon \right) \vec{\varphi}(x) dx \leq C |\rho^\varepsilon|_{\Omega} \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)}^2 \|\vec{\varphi}\|_{L_4(\Omega)},$$

we find

$$\int_{\Omega} \nabla p^\varepsilon \vec{\varphi}(x) dx \leq C \left(1 + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)} \right) \|\vec{\varphi}\|_{W_2^1(\Omega)}.$$

It follows from this

$$\|\nabla p^\varepsilon\|_{W_2^{-1}(\Omega)} \leq C \left(1 + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)} \right).$$

This is due to the condition $\int_{\Omega} p^\varepsilon dx = 0$ and the inequality

$$\|p^\varepsilon\|_{L_2(\Omega)} \leq C \|\nabla p^\varepsilon\|_{W_2^{-1}(\Omega)},$$

allows you to set a rating

$$\|p^\varepsilon\|_{L_2(\Omega)} \leq C \left(1 + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)} \right), \text{ from which follows (37).}$$

The proof of the solvability of problem (33) – (36) is carried out by the Galerkin method based on (37).

Corollary. Any generalized solution of problem (33) – (36) as $\varepsilon \rightarrow 0$ becomes a generalized solution of problem (1) – (4).

Indeed, by virtue of (37), from the sequence $\{\vec{v}^\varepsilon, p^\varepsilon, \rho^\varepsilon\}$ one can select a subsequence that converges as $\varepsilon \rightarrow 0$ in the following sense:

$$\vec{v}^\varepsilon \rightarrow \vec{v} \text{ weakly in } W_2^1(\Omega) \text{ and strongly in } L_6(\Omega),$$

$$\rho^\varepsilon \rightarrow \rho \text{ weakly in } W_2^2(\Omega),$$

$$p^\varepsilon \rightarrow p \text{ weakly in } L_2(\Omega).$$

These properties of the sequence $\{\vec{v}^\varepsilon, p^\varepsilon, \rho^\varepsilon\}$ allow us to move to the redistribution as $\varepsilon \rightarrow 0$ in the integral identities for $\vec{v}^\varepsilon, p^\varepsilon, \rho^\varepsilon$, similar to those given in Definition 2. We will show how the transition is carried out in the most complex nonlinear terms

$$\begin{aligned} & |(\rho^\varepsilon(\vec{v}^\varepsilon \cdot \nabla))\vec{v}^\varepsilon, \vec{\psi}) - (\rho(\vec{v} \cdot \nabla)\vec{v}, \vec{\psi})| = \\ & = |(\rho^\varepsilon - \rho, (\vec{v}^\varepsilon \cdot \nabla)\vec{v}^\varepsilon, \vec{\psi}) + (\rho[(\vec{v}^\varepsilon - \vec{v}) \cdot \nabla]\vec{v}^\varepsilon, \vec{\psi}) + (\rho(\vec{v} \cdot \nabla)(\vec{v}^\varepsilon - \vec{v}), \vec{\psi})| \leq \\ & \leq |\rho^\varepsilon - \rho| \|\vec{v}^\varepsilon\|_{L_4(\Omega)} \|\vec{v}^\varepsilon\|_{W_2^1(\Omega)} \|\vec{\psi}\|_{L_4(\Omega)} + |\rho| \|\vec{v}^\varepsilon - \vec{v}\|_{L_4(\Omega)} \|\vec{v}^\varepsilon\|_{W_2^1(\Omega)} \|\vec{\psi}\|_{L_4(\Omega)} + \\ & \quad + |\rho(\vec{v} \cdot \nabla)(\vec{v}^\varepsilon - \vec{v}), \vec{\psi}|. \end{aligned}$$

As $\varepsilon \rightarrow 0$, the first term tends to zero due to the strong convergence of ρ^ε and ρ (which follows from the compactness of the embedding of $W_2^2(\Omega)$ into $C(\Omega)$), the second term due to the strong convergence of \vec{v}^ε to \vec{v} in $L_4(\Omega)$. We integrate the third term by parts

$$\int_{\Omega} \rho(\vec{v} \cdot \nabla)[\vec{v}^\varepsilon - \vec{v}]\vec{\psi} dx = - \int_{\Omega} [\vec{v}^\varepsilon - \vec{v}](\vec{v} \cdot \nabla)(\rho\vec{\psi}) dx.$$

Obviously, the inequality is true

$$\begin{aligned} & |(\rho(\vec{v} \cdot \nabla)(\vec{v}^\varepsilon - \vec{v}), \vec{\psi})| \leq \|\vec{v}^\varepsilon - \vec{v}\|_{L_4(\Omega)} \|\vec{v}\|_{L_4(\Omega)} \times \\ & \quad \times (|\rho| \|\vec{\psi}\|_{W_2^1(\Omega)} + \|\vec{\psi}\|_{L_4(\Omega)} \|\rho\|_{W_4^1(\Omega)}), \end{aligned}$$

from which it follows that

$$(\rho^\varepsilon(\vec{v}^\varepsilon \cdot \nabla))\vec{v}^\varepsilon, \vec{\psi}) \rightarrow (\rho(\vec{v} \cdot \nabla)\vec{v}, \vec{\psi}).$$

In the remaining terms of the integral identity, the limit transition is carried out similarly.

Theorem 5. If $\vec{f}(x) \in L_2(\Omega)$, $\rho_0(x) \in W_2^{3/2}(\partial\Omega)$, $\partial\Omega \in C^2$, then the generalized solution of problem (33) – (36) becomes strong and the estimate

$$\|\vec{v}^\varepsilon\|_{W_2^1(\Omega)} + \|\rho^\varepsilon\|_{W_2^2(\Omega)} + \|\nabla p^\varepsilon\|_{L_2(\Omega)} \leq C_\varepsilon,$$

where $C_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. As was seen above, the following estimate holds:

$$\varepsilon^{1/2} \|\nabla p\|_{L_4(\Omega)} \leq C \left(\|\vec{f}\|_{W_2^{-1}(\Omega)}, \|\rho_0\|_{W_2^{3/2}(\partial\Omega)} \right).$$

Let us write equation (33) in the form

$$\nu \Delta \vec{v}^\varepsilon = \nabla p^\varepsilon - \lambda(\nabla \rho^\varepsilon \cdot \nabla) \vec{v}^\varepsilon + \rho^\varepsilon (\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon - \rho^\varepsilon \vec{f} + \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon,$$

From the theory of boundary value problems for elliptic equations it follows that $\vec{v}^\varepsilon(x) \in W_2^2(\Omega) \cap \dot{W}_2^1(\Omega)$ and the inequality holds

$$\|\vec{v}^\varepsilon\|_{W_2^2(\Omega)} \leq C \left(\|\nabla \rho\|_{L_4(\Omega)} \|\nabla \vec{v}^\varepsilon\|_{L_4(\Omega)} + M \|\vec{v}^\varepsilon\|_{L_6(\Omega)} \|\nabla \vec{v}^\varepsilon\|_{L_3(\Omega)} + M \|\vec{f}\|_{L_2(\Omega)} \right).$$

The following estimate follows from this:

$$\varepsilon^{1/2} \|\vec{v}^\varepsilon\|_{W_2^2(\Omega)W_2^1(\Omega)} \leq C \left(\|\vec{f}\|_{W_2^{-1}(\Omega)}, \|\rho_0\|_{W_2^{3/2}(\partial\Omega)} \right).$$

This is what needed to be proven.

Theorem 6. Let $\vec{f}(x) \in W_2^1(\Omega)$, $\rho_0(x) \in W_2^{3/2}(\partial\Omega)$, $\partial\Omega \in C^3$, and $\|\vec{f}\|_{W_2^{-1}(\Omega)}, M$ be sufficiently small. Then the following estimate holds:

$$\|\vec{v}^\varepsilon - \vec{v}\|_{W_2^1(\Omega)}^2 + \|p^\varepsilon - p\|_{L_2(\Omega)}^2 + \|\rho^\varepsilon - \rho\|_{W_2^2(\Omega)}^2 \leq C_4 \varepsilon^{3/2}, \quad (38)$$

$$\|\vec{v}^\varepsilon - \vec{v}\|_{W_2^2(\Omega)}^2 + \|p^\varepsilon - p\|_{W_2^1(\Omega)}^2 \leq C_5 \varepsilon^{1/2}, \quad (39)$$

Proof. First of all, we note that from the general theory of boundary value problems for the Navier-Stokes equations it follows that if $\vec{f}(x) \in W_2^1(\Omega)$, then $\vec{v}_{xxx} \in L_2(\Omega)$ and $p_{xx} \in L_2(\Omega)$, i.e. $\vec{v}_x, \vec{v} \in C(\bar{\Omega})$.

We represent the solution of problem (33) – (36) as the sum

$$\vec{v}^\varepsilon = \vec{v} + \varepsilon \vec{w}, p^\varepsilon = p + \varepsilon \pi, \rho^\varepsilon = \rho + \varepsilon \eta.$$

Then the functions \vec{w}, π, η can be considered as the solution of the following boundary value problem

$$\eta(\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon + \rho[(\vec{v} \cdot \nabla) \vec{w} + (\vec{w} \cdot \nabla) \vec{v}^\varepsilon] = \nu \Delta \vec{w} - \nabla \pi + \quad (40)$$

$$+ \lambda[(\nabla \rho \cdot \nabla) \vec{w} + (\nabla \eta \cdot \nabla) \vec{v}^\varepsilon] + \eta \vec{f} - \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{w},$$

$$\operatorname{div} \vec{w} = \varepsilon \Delta \pi - \varepsilon \pi + \Delta p - p, \quad (41)$$

$$(\vec{v} \cdot \nabla) \eta + (\vec{w} \cdot \nabla) \rho^\varepsilon = \lambda \Delta \eta, \quad (42)$$

$$\vec{w}|_{\partial\Omega} = 0, \frac{\partial \pi}{\partial \eta} \Big|_{\partial\Omega} = -\frac{1}{\varepsilon} \frac{\partial p}{\partial \eta} \Big|_{\partial\Omega}, \eta|_{\partial\Omega} = 0. \quad (43)$$

Multiply (40) by \vec{w} in $L_2(\Omega)$ and integrate by parts

$$\begin{aligned} \nu \|\vec{w}\|_{L_2(\Omega)}^2 - \int_{\Omega} \pi \operatorname{div} \vec{w} \, dx &= \int_{\Omega} \left[\eta(\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon + \rho((\vec{v} \cdot \nabla) \vec{w} + (\vec{w} \cdot \nabla) \vec{v}^\varepsilon) + \right. \\ &\left. + \lambda((\nabla \rho \cdot \nabla) \vec{w} + (\nabla \eta \cdot \nabla) \vec{v}^\varepsilon) + \eta \vec{f} - \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{w} \right] \vec{w} \, dx. \end{aligned}$$

The following relations are valid:

$$\begin{aligned}
 - \int_{\Omega} \rho(\vec{v} \cdot \nabla) \vec{w} \cdot \vec{w} dx &= - \frac{1}{2} \int_{\Omega} \rho(\vec{v} \cdot \nabla)(\vec{w})^2 dx = \frac{1}{2} \int_{\Omega} \vec{w}^2(\vec{v} \cdot \nabla) \rho dx = \\
 &= \frac{1}{2} \int_{\Omega} \vec{w}^2 \lambda \Delta \rho dx = - \lambda \int_{\Omega} (\nabla \rho \cdot \nabla) \vec{w} \cdot \vec{w} dx, \\
 - \int_{\Omega} \pi \operatorname{div} \vec{w} dx &= - \int_{\Omega} \pi(\varepsilon \Delta \pi - \varepsilon \pi + \Delta p - p) dx \\
 &= \varepsilon \|\nabla \pi\|_{L_2(\Omega)}^2 - \varepsilon \int_{\partial \Omega} \pi \frac{\partial \pi}{\partial \eta} d(\partial \Omega) + \varepsilon \|\pi\|_{L_2(\Omega)}^2 - \\
 - \int_{\Omega} \pi(\Delta p - p) dx &\leq \varepsilon \|\nabla \pi\|_{L_2(\Omega)}^2 + \varepsilon \|\pi\|_{L_2(\Omega)}^2 + \|\pi\|_{L_2(\Omega)} \left\| \frac{\partial \pi}{\partial \eta} \right\|_{L_2(\Omega)} + \\
 &+ \|\pi\|_{L_2(\Omega)} (\|\Delta p\|_{L_2(\Omega)} + \|p\|_{L_2(\Omega)}), \\
 \int_{\Omega} \left[-\eta(\vec{v}^\varepsilon \cdot \nabla) \vec{v}^\varepsilon - \rho(\vec{w} \cdot \nabla) \vec{v}^\varepsilon + \lambda(\nabla \eta \cdot \nabla) \vec{v}^\varepsilon - \frac{1}{2} \rho^\varepsilon \vec{v}^\varepsilon \operatorname{div} \vec{v}^\varepsilon \right] \vec{w} dx &\leq \\
 &\leq C(\|\eta\|_{L_6(\Omega)} \|\vec{v}^\varepsilon\|_{L_6(\Omega)} \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\vec{w}\|_{L_6(\Omega)} + M \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\vec{w}\|_{L_4(\Omega)}^2 + \\
 &+ \|\nabla \eta\|_{L_4(\Omega)} \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\vec{w}\|_{L_4(\Omega)} + M \|\vec{v}^\varepsilon\|_{L_4(\Omega)} \|\vec{w}\|_{L_4(\Omega)} \|\nabla \vec{w}\|_{L_4(\Omega)}).
 \end{aligned}$$

From (42) we obtain

$$\lambda \|\nabla \eta\|_{L_2(\Omega)}^2 \leq C \|\rho^\varepsilon(\vec{w} \cdot \nabla)\|_{L_2(\Omega)}^2 \leq C \|\vec{w}\|_{L_4(\Omega)}^2 \|\nabla \rho^\varepsilon\|_{L_4(\Omega)}^2 \leq C \|\nabla \vec{w}\|_{L_2(\Omega)} \|\nabla \rho^\varepsilon\|_{L_2(\Omega)}.$$

Let us estimate the following quantities

$$\begin{aligned}
 \|\pi\|_{L_2(\Omega)} &\leq C \|\nabla \pi\|_{W_2^1(\Omega)} \leq \\
 &\leq C \left\{ \|\nabla \vec{w}\|_{L_2(\Omega)} \left[1 + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\partial \Omega)} \right] + \|\nabla \eta\|_{L_2(\Omega)} \left[1 + \|\vec{f}\|_{W_2^{-1}(\Omega)} \right] \right\}. \\
 \|\pi\|_{L_2(\partial \Omega)} \cdot A &\leq C \|\nabla \pi\|_{L_2(\Omega)}^{1/2} \cdot A \leq C(\varepsilon \|\nabla \pi\|_{L_2(\Omega)}^2)^{\frac{1}{4}} \cdot (\|\pi\|_{L_2(\Omega)}^2)^{\frac{1}{4}} \cdot \varepsilon^{-\frac{1}{4}} \cdot A \leq \\
 &\leq \frac{\delta_1}{4} \varepsilon \|\nabla \pi\|_{L_2(\Omega)}^2 + \frac{\delta_1}{4} \|\pi\|_{L_2(\Omega)}^2 + \frac{\delta_1^{-1}}{2} C \cdot \varepsilon^{-\frac{1}{2}} \cdot A^2 \\
 \|\eta\|_{L_6(\Omega)} \|\vec{v}^\varepsilon\|_{L_6(\Omega)} \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\vec{w}\|_{L_6(\Omega)} &+ M \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\vec{w}\|_{L_4(\Omega)}^2 + \\
 + \|\nabla \eta\|_{L_4(\Omega)} \|\nabla \vec{v}^\varepsilon\|_{L_2(\Omega)} \|\vec{w}\|_{L_4(\Omega)} &+ M \|\vec{v}^\varepsilon\|_{L_4(\Omega)} \|\nabla \vec{w}\|_{L_2(\Omega)} \|\vec{w}\|_{L_2(\Omega)} \leq
 \end{aligned}$$

$$\begin{aligned} &\leq \|\nabla\eta\|_{L_2(\Omega)}\|\nabla\vec{v}^\varepsilon\|_{L_2(\Omega)}^2\|\nabla\vec{w}\|_{L_2(\Omega)} + M\|\nabla\vec{v}^\varepsilon\|_{L_2(\Omega)}\|\nabla\vec{w}\|_{L_2(\Omega)}^2 + \\ &+ \|\nabla\eta\|_{L_2(\Omega)}\|\nabla\vec{v}^\varepsilon\|_{L_2(\Omega)}\|\nabla\vec{w}\|_{L_2(\Omega)} + M\|\nabla\vec{v}^\varepsilon\|_{L_2(\Omega)}\|\nabla\vec{w}\|_{L_2(\Omega)}^2 \leq \\ &\leq C\|\nabla\vec{w}\|_{L_2(\Omega)}^2\|\vec{f}\|_{W_2^{-1}(\Omega)}^2 \left[1 + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)}\right] \end{aligned}$$

Using the above inequalities, for a sufficiently small δ_1 we obtain

$$\begin{aligned} &\|\nabla\vec{w}\|_{L_2(\Omega)} \left\{ \nu - C\|\vec{f}\|_{W_2^{-1}(\Omega)}^2 \left[1 + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)}\right] \right\} + \\ &+ \varepsilon\|\nabla\pi\|_{L_2(\Omega)}^2 + \varepsilon\|\pi\|_{L_2(\Omega)}^2 \leq C \left(\|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + C\varepsilon^{-\frac{1}{2}} \right) \end{aligned}$$

Let $\vec{f}(x)$ and $\rho_0(x)$ be such that

$$\nu - C\|\vec{f}\|_{W_2^{-1}(\Omega)}^2 \left[1 + \|\vec{f}\|_{W_2^{-1}(\Omega)}^2 + \|\rho_0\|_{W_2^{3/2}(\partial\Omega)}\right] \geq \nu_0 > 0$$

Then the following estimate takes place:

$$\|\nabla\eta\|_{L_2(\Omega)}^2 + \nu_0\|\nabla\vec{w}\|_{L_2(\Omega)}^2 + \varepsilon\|\nabla\pi\|_{L_2(\Omega)}^2 + \varepsilon\|\pi\|_{L_2(\Omega)}^2 \leq C_6\varepsilon^{-\frac{1}{2}},$$

It follows from this

$$\|\vec{v}^\varepsilon - \vec{v}\|_{W_2^1(\Omega)}^2 + \|p^\varepsilon - p\|_{L_2(\Omega)}^2 + \|\rho^\varepsilon - \rho\|_{W_2^2(\Omega)}^2 \leq C_6\varepsilon^2\varepsilon^{-\frac{1}{2}} = C_6\varepsilon^{\frac{3}{2}},$$

$$\|\vec{v}^\varepsilon - \vec{v}\|_{W_2^1(\Omega)}^2 + \|p^\varepsilon - p\|_{L_2(\Omega)}^2 \leq C_7\varepsilon^{\frac{1}{2}}.$$

Theorem 6 is proven.

Application Example

Stationary Heat Transfer in Climate Systems

One of the practical applications of the developed model is simulating stationary heat transfer in multilayer geophysical systems, such as the ocean–ice–atmosphere interface. In such problems, physical heterogeneity arises due to abrupt changes in thermal conductivity between layers (e.g., ice and water), while modified boundary conditions account for weak but non-negligible heat exchange at the interfaces. In this context, the small parameter ε represents the thickness of transition zones or an effective heat exchange coefficient, allowing the model to account for the influence of thin snow layers on top of ice or turbulent atmospheric effects.

The stationary formulation enables the evaluation of stable temperature fields within the ice sheet under given boundary conditions. Modified boundary conditions are particularly relevant for scenarios where the ice–air interface is not ideally insulated but permits a weak heat flux.

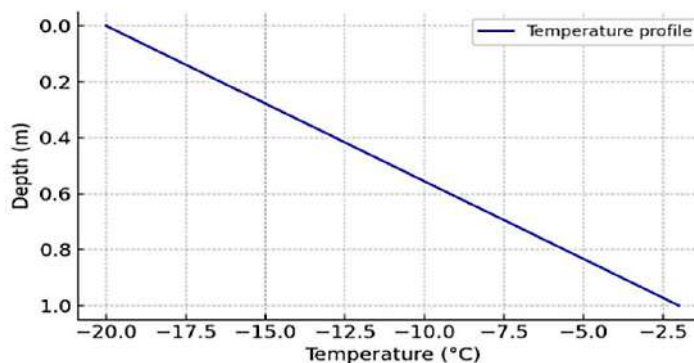


Figure 1. Stationary temperature distribution in the ice cover under given boundary conditions

The figure below illustrates a simplified numerical example: the stationary temperature distribution across a 1-meter-thick ice layer. The top boundary (ice–air interface) is set at $-20\text{ }^{\circ}\text{C}$, and the bottom (ice–water interface) at $-2\text{ }^{\circ}\text{C}$. The result shows a linear temperature gradient, typical of steady-state thermal conduction in homogeneous media, but applicable here with consideration of boundary modifications. Such models are essential for long-term climate predictions, simulations of ice melt dynamics, and energy balance studies in polar regions.

Results of the study

In the course of the theoretical study, a stationary diffusion problem in an inhomogeneous medium with small parameters and modified boundary conditions was considered. Three different mathematical approximations of the original model were constructed and rigorously analyzed. At each stage, the correctness of the formulation was substantiated, the existence of solutions was proven, and a priori estimates were obtained that do not depend on the values of the small parameters.

For the original problem with fixed boundary conditions, it was shown that when the conditions of the boundedness of the boundary density function are met, there is at least one strong solution. Using the maximum principle, Young's inequalities, and energy estimates, inequalities were derived that enable the establishment of solution stability.

A transition was made to approximate problems that include small parameters reflecting the features of the boundary and the inhomogeneity of the medium. An asymptotic expansion of the solution in powers of the small parameters was constructed. It was found that as the parameters tend to zero, the solutions of the approximated problems weakly converge to the solution of the original model. Moreover, if the conditions of sufficient smallness of the parameters are met, it is proved that the expansion preserves the accuracy estimate and allows analytical control of the residual terms.

The concept of a generalized solution is also introduced, and it is proved that it exists and coincides with the weak limit of approximations. The conditions under which the generalized solution becomes strong are established. This confirms the correctness of the constructed approach both in theoretical and applied aspects, including heat transfer problems in multilayer media, where the physical model requires taking into account complex boundary interactions.

Quantitative estimates of the model accuracy

As part of the work, a priori estimates of solutions for various forms of problem formulation were obtained - both in the original model and in its approximated versions with small parameters. These estimates enabled us to quantitatively characterize the behavior of solutions as the parameters that model the medium's inhomogeneities and the features of the boundary interaction were changed. In particular, based on energy inequalities and the application of the Galerkin method, upper bounds of the norms of solutions were established that do not depend on small parameters. Such estimates (of the form (7), (9), (20), (37)) serve as a basis for proving the stability of solutions, as well as for analyzing their convergence as the parameters tend to zero.

Error estimates between approximate and exact solutions were obtained. It was found that for sufficiently small parameters, the expansion presented as a sum of the main and correcting terms ensures an accuracy of order $O(\varepsilon^2 + \delta^2)$ in parameters. This allows one to control the approximation error at all stages and apply the model in problems requiring high reliability of the numerical forecast.

Based on the obtained estimates, it was also demonstrated that a limit exists as the parameters tend to zero, and the limit solution satisfies the original problem in a generalized sense. This result is confirmed strictly through sequences of approximations that weakly and strongly converge in the corresponding functional spaces, as described in the proofs of Theorems 4–6.

Thus, the quantitative characteristics of the model not only support its theoretical validity but also allow it to be effectively applied in practical problems of diffusion analysis in inhomogeneous media with thin boundaries and transition layers. These estimates create a basis for developing robust numerical methods adapted to the parametric structure of the problem.

Discussion

The results of this study emphasize the importance of modifying boundary conditions in mathematical modeling of diffusion processes in heterogeneous media with small parameters. Classical approaches based on rigid boundary conditions, as a rule, do not take into account the influence of real transient processes at the boundary of the medium. In contrast, the proposed model allows for a flexible description of boundary phenomena that arise, for example, due to weak interactions between physical layers or abrupt changes in material characteristics.

Analytical and numerical results demonstrate that the model exhibits high stability and predictability when varying the parameters responsible for the medium's heterogeneity and boundary diffusion. At the same time, the obtained approximations retain accuracy. They are consistent with the physical nature of the processes, making the model applicable to a wide range of problems, from thermal physics and climatology to engineering systems with a multilayer structure.

Of particular value is the possibility of conducting a rigorous analysis of the influence of boundary conditions on the structure of solutions, which is of key importance in situations where instability at the boundary can significantly affect the global behavior of the system. The justification of convergence and the availability of quantitative estimates confirm that the model can serve as a reliable basis for subsequent numerical implementation.

Conclusion

The paper constructs and analyzes a model of stationary diffusion in an inhomogeneous medium, taking into account small parameters and modified boundary conditions. A mathematically rigorous statement of the problem is formulated, the existence and uniqueness of a solution are proven, a priori estimates are obtained, and an asymptotic analysis of the solutions is performed. A technique for constructing approximate solutions with accuracy control is proposed.

The application of the model to the problem of temperature distribution in ice cover made it possible to demonstrate its applied value and confirm the physical validity of the obtained solutions. The conducted modeling showed that taking into account boundary inhomogeneities significantly affects the field structure in the vicinity of the boundary. At the same time, in the central region, the solution remains stable and regular.

Thus, it can be argued that the developed model is mathematically correct, stable to parametric disturbances, and applicable to a broad class of problems. Its further development can be associated with taking into account nonlinear effects, the transition to non-stationary modes, and the construction of adaptive numerical algorithms for practical application in engineering and natural systems.

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