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SOLVING SYSTEMS OF SECOND-ORDER DIFFERENTIAL EQUATIONS FOR SOME MULTIDIMENSIONAL HYPERGEOMETRIC FUNCTIONS

Abstract

In this paper, we study the problem of finding solutions to a system of second-order partial differential equations related to hypergeometric functions of four variables F_{16}, F_{18}, F_{19}, F_{20} and F_{31}. Currently, mathematical modeling of physical processes and forecasting their dynamics are becoming increasingly important in science and technology. Most of these processes are described by differential equations or their systems, but only a few of them, those that reflect real physical phenomena, allow analytical solutions expressed in terms of elementary functions. This fact makes it particularly relevant to study more general classes of functions, in particular hypergeometric functions of many variables. Hypergeometric functions are solutions of certain systems of differential equations and belong to the category of special or transcendental functions. Due to their properties, they are widely used in solving multidimensional problems, and the study of their analytical structure is important both for the development of theoretical mathematics and for practical calculations. Within the framework of this study, linearly independent solutions of these systems were constructed for the functions under consideration, as well as their structural features and main characteristics were analyzed. The results obtained form the theoretical basis for further study of hypergeometric functions of a higher number of variables and open up prospects for their effective use in solving applied problems. The work contributes to the development of the theory of special functions and offers new approaches to solving fundamental problems of mathematical physics and applied mathematics.

Keywords: hypergeometric function of four variables, hypergeometric type differential equation, linearly independent solutions of the system.

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КЕЙБІР КӨП ӨЛШЕМДІ ГИПЕРГЕОМЕТРИЯЛЫҚ ФУНКЦИЯЛАР ҮШІН ЕКІНШІ РЕТТІ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ЖҮЙЕСІН ШЕШУ

Аңдатпа

Бұл мақалада төрт айнымалы гипергеометриялық F_{16}, F_{18}, F_{19}, F_{20} және F_{31} және функциялары үшін дербес туындылы екінші ретті дифференциалдық теңдеулер жүйесін шешу мәселесі қарастырылады. Қазіргі таңда физикалық процестерді математикалық модельдеу және олардың нәтижелерін болжау мәселелері өзекті болып отыр. Мұндай модельдер көбінесе дифференциалдық теңдеулер немесе олардың жүйелері арқылы сипатталады. Алайда, нақты физикалық құбылыстарды сипаттайтын бұл теңдеулердің тек аз бөлігі ғана элементар функциялармен өрнектелетін шешімдерге ие болады. Осы себепті жаңа функцияларды, соның ішінде көп айнымалы гипергеометриялық функцияларды зерттеу ерекше маңызға ие. Гипергеометриялық функциялар дифференциалдық теңдеулер жүйесінің шешімдері ретінде анықталып, арнайы немесе трансценденттік функциялар класына жатады. Бұл функциялар көп айнымалы есептерде кеңінен қолданылады және олардың аналитикалық қасиеттерін зерттеу теориялық қана емес, практикалық тұрғыдан да маңызды. Зерттеу барысында аталған функциялар үшін қарастырылған дифференциалдық жүйелердің сызықтық тәуелсіз шешімдері табылып, олардың құрылымдық ерекшеліктері мен қасиеттері талданды. Алынған нәтижелер гипергеометриялық функциялардың жоғары өлшемді кеңейтулерін әрі қарай зерттеуге негіз болады және оларды қолданбалы есептерге тиімді қолдануға жол ашады. Бұл зерттеу арнайы функциялар теориясын тереңдетіп, математикалық физика мен қолданбалы математиканың маңызды мәселелерін шешуде жаңа бағыттар ұсынады.

Түйін сөздер: төрт айнымалы гипергеометриялық функция, екінші ретті дербес туындылы дифференциалдық теңдеу, жүйенің сызықтық тәуелсіз шешімдері.

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РЕШЕНИЕ СИСТЕМ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА ДЛЯ НЕКОТОРЫХ МНОГОМЕРНЫХ ГИПЕРГЕОМЕТРИЧЕСКИХ ФУНКЦИЙ

Аннотация

В данном исследовании рассматривается решение системы дифференциальных уравнений в частных производных второго порядка, соответствующих гипергеометрическим функциям с четырьмя переменными F_{16}, F_{18}, $F_{19,}$ F_{20} и F_{31}. В современной науке математическое моделирование физических процессов и прогнозирование их поведения играют решающую роль. Многие такие процессы описываются дифференциальными уравнениями или системами уравнений; однако лишь ограниченное число из них те, которые описывают реальные физические явления, могут быть решены в терминах элементарных функций. Это ограничение подчеркивает важность исследования более широких классов функций, в частности гипергеометрических функций нескольких переменных. Гипергеометрические функции возникают как решения конкретных систем дифференциальных уравнений и относятся к классу специальных или трансцендентных функций. Они находят широкое применение в задачах, связанных с несколькими переменными, и понимание их аналитической структуры важно как для теоретических, так и для практических целей. В настоящей работе были построены линейно независимые решения исследуемых систем для вышеупомянутых функций, а также исследована их внутренняя структура и ключевые свойства. Полученные результаты создают основу для разработки многомерных обобщений гипергеометрических функций и указывают на новые возможности их практического использования при решении прикладных задач. Это исследование обогащает теорию специальных функций и открывает новые перспективы для решения фундаментальных задач математической физики и прикладной математики.

Ключевые слова: гипергеометрическая функция четырёх переменных, дифференциальное уравнение гипергеометрического типа, линейно-независимые решения системы.

Introduction

In such fields of science as thermal conductivity and mechanics, electromagnetic oscillations, aerodynamics, quantum mechanics and potential theory, transcendental functions, the so-called hypergeometric series, are widely used in solving applied problems [1, 2]. These functions, traditionally called special functions of mathematical physics, naturally arise when solving differential equations analyzed by means of harmonic analysis. An in-depth study of multidimensional hypergeometric functions is of particular importance due to their wide application possibilities in various fields of science. The need to solve complex problems of quantum chemistry [3, 4], the study of superstring theory [5], as well as numerous problems of gas dynamics have stimulated the active development of the theory of special functions of one and several variables. In particular, in gas dynamics, many formulations are reduced to the study of independent second-order differential equations, the solutions of which are expressed in terms of hypergeometric functions of many variables [6, 7]. In this regard, when studying boundary value problems for such equations, it becomes necessary to construct systems of hypergeometric functions and find their explicit linearly independent solutions [3, 8-10].

A significant contribution to the development of this field was made by Exton [11], who proposed a four-dimensional hypergeometric function of class K_1-K_21. Subsequently, Sharma and Parihar [12] extended these results by defining a family of hypergeometric functions of four variables with 83 different parametric variants.

Each four-dimensional hypergeometric function has the following form:

$$F^{(4)}(\cdot)=\sum_{m,n,p,q}^{\infty}\Delta(m,n,p,q)\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{1}$$

here $\Delta(m,n,p,q)$ is some sequence of complex parameters, and each function has twelve parameters. Consider the hypergeometric functions $F_{16}^{(4)}$, $F_{18}^{(4)}$, $F_{19}^{(4)}$, $F_{20}^{(4)}$, $F_{31}^{(4)}$:

$$F_{16}^{(4)}(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+q}(b_2)_p}{(c_1)_m(c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{2}$$

$$F_{18}^{(4)}(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_{p+q}}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{3}$$

$$F_{19}^{(4)}(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_{p+q}}{(c_1)_m(c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{4}$$

$$F_{20}^{(4)}(a_1,a_2,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}, \tag{5}$$

$$F_{31}^{(4)}(a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_3;c_1,c_1,c_2,c_3;x,y,z,t)=$$
$$\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n)}(a_2)_{p+q}(b_1)_{m+q}(b_2)_p(b_3)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}. \tag{6}$$

Methodology

This paper presents methodological approaches to solving a system of second-order differential equations with partial derivatives describing hypergeometric functions of four variables of the form $F(a,b,c,d;x,y,z,t)$. Since such systems define special functions depending on multivariate arguments, their solutions possess a complex analytical structure. The results demonstrate that these functions, being solutions of special differential systems, can serve as an effective tool for solving a wide range of problems in physics and applied mathematics. The analysis provides a basis for further study of their symmetries, analytical properties, and structural features.

The solutions are interpreted as multivariate generalizations of hypergeometric functions and compared with classical Gauss, Appell, and Lauricella functions. However, the study has certain limitations. In particular, issues of convergence of the obtained solutions and their stability under boundary conditions require a more detailed analysis. Additional research is also needed to verify the applicability of these solutions within the framework of real physical models. All this opens up promising avenues for a deeper understanding of the internal structure of the theory of special functions of many variables and the search for new analytical forms.

Solution of a system of independent differential equations of functions $F_{16}^{(4)}$, $F_{18}^{(4)}$, $F_{19}^{(4)}$, $F_{20}^{(4)}$, $F_{31}^{(4)}$

Let's consider the hypergeometric function (2) $F_{16}^{(4)}$:

$$F_{16}^{(4)}(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+q}(b_2)_p}{(c_1)_m(c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

where c ≠ 0, -1, -2, -3, ….; here i=1, …., 3.

First we use the Kampe de Feriet series [13],

$$\begin{aligned}
A_{m,n,p,q}&=\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+q}(b_2)_p}{(c_1)_m(c_2)_n(c_3)_{p+q}\,m!n!p!q!},\\
A_{m+1,n,p,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+q+1}(b_2)_p}{(c_1)_{m+1}(c_2)_n(c_3)_{p+q}(m+1)!n!p!q!},\\
A_{m,n+1,p,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+q+1}(b_2)_p}{(c_1)_m(c_2)_{n+1}(c_3)_{p+q}\,m!(n+1)!p!q!},\\
A_{m,n,p+1,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+q}(b_2)_{p+1}}{(c_1)_m(c_2)_n(c_3)_{p+q+1}\,m!n!(p+1)!q!},\\
A_{m,n,p,q+1}&=\frac{(a_1)_{m+n+p}(a_2)_{q+1}(b_1)_{m+n+q+1}(b_2)_p}{(c_1)_m(c_2)_n(c_3)_{p+q+1}\,m!n!p!(q+1)!}.
\end{aligned}\tag{7}$$

Here $A_{m,n,p,q}$ are the coefficients of the Kampe de Feriet series, and $a_1, a_2, b_1,\ b_2, c_1, c_2, c_3$ are the parameters [14].

$$\begin{aligned}
A_{m+1,n,p,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+q+1}(b_2)_p}{(c_1)_{m+1}(c_2)_n(c_3)_{p+q}(m+1)!n!p!q!}\frac{(c_1)_m(c_2)_n(c_3)_{p+q}\,m!n!p!q!}{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+q}(b_2)_p},\\
A_{m,n+1,p,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+q+1}(b_2)_p}{(c_1)_m(c_2)_{n+1}(c_3)_{p+q}\,m!(n+1)!p!q!}\frac{(c_1)_m(c_2)_n(c_3)_{p+q}\,m!n!p!q!}{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+q}(b_2)_p},\\
A_{m,n,p+1,q}&=\frac{(a_1)_{m+n+p+1}(a_2)_q(b_1)_{m+n+q}(b_2)_{p+1}}{(c_1)_m(c_2)_n(c_3)_{p+q+1}\,m!n!(p+1)!q!}\frac{(c_1)_m(c_2)_n(c_3)_{p+q}\,m!n!p!q!}{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+q}(b_2)_p},\\
A_{m,n,p,q+1}&=\frac{(a_1)_{m+n+p}(a_2)_{q+1}(b_1)_{m+n+q+1}(b_2)_p}{(c_1)_m(c_2)_n(c_3)_{p+q+1}\,m!n!p!(q+1)!}\frac{(c_1)_m(c_2)_n(c_3)_{p+q}\,m!n!p!q!}{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n+q}(b_2)_p}.
\end{aligned}\tag{8}$$

From the series (8) we obtain a one-sided recurrence relation (9) of coefficients depending on m, n, p, q:

$$\begin{aligned} &\frac{A_{m+1,n,p,q}}{A_{m,n,p,q}} = \frac{(a_1+m+n+p)(b_1+m+n+q)}{(c_1+m)(m+1)}, \\ &\frac{A_{m,n+1,p,q}}{A_{m,n,p,q}} = \frac{(a_1+m+n+p)(b_1+m+n+q)}{(c_2+n)(n+1)}, \\ &\frac{A_{m,n,p+1,q}}{A_{m,n,p,q}} = \frac{(a_1+m+n+p)(b_2+p)}{(c_3+p+q)(p+1)}, \\ &\frac{A_{m,n,p,q+1}}{A_{m,n,p,q}} = \frac{(a_2+q)(b_1+m+n+q)}{(c_3+p+q)(q+1)}. \end{aligned} \tag{9}$$

Where $m = x\frac{\partial}{\partial x}, n = y\frac{\partial}{\partial y}, p = z\frac{\partial}{\partial z}, q = t\frac{\partial}{\partial t}$ are defined as differential operators.

According to the theory of multidimensional hypergeometric functions, the system of independent differential equations for the hypergeometric function $F_{16}^{(4)}$ is given as follows:

$$\begin{aligned} &\left(c_1 + x\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x}+1\right)x^{-1}u - \left(a_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(b_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right)u = 0, \\ &\left(c_2 + y\frac{\partial}{\partial y}\right)\left(y\frac{\partial}{\partial y}+1\right)y^{-1}u - \left(a_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(b_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right)u = 0, \\ &\left(c_3 + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right)\left(z\frac{\partial}{\partial z}+1\right)z^{-1}u - \left(a_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(b_2 + z\frac{\partial}{\partial z}\right)u = 0, \\ &\left(c_3 + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}\right)\left(t\frac{\partial}{\partial t}+1\right)t^{-1}u - \left(a_2 + t\frac{\partial}{\partial t}\right)\left(b_1 + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t}\right)u = 0. \end{aligned} \tag{10}$$

Applying simple calculations to (10), we obtain the following system of independent second order differential equations:

$$\begin{cases} x(1-x)u_{xx} - 2xyu_{xy} - xtu_{xt} - xzu_{xz} - y^2u_{yy} - ytu_{yt} - yzu_{yz} - ztu_{zt} + \left[c_1 - (a_1+b_1+1)x\right]u_x \\ -(a_1+b_1+1)yu_y - a_1tu_t - b_1zu_z - a_1b_1u = 0, \\ y(1-y)u_{yy} - x^2u_{xx} - 2xyu_{xy} - xzu_{xz} - yzu_{yz} - ytu_{yt} - xtu_{xt} - ztu_{zt} - (a_1+b_1+1)xu_x \\ +\left[c_2 - (a_1+b_1+1)y\right]u_y - a_1tu_t - b_1zu_z - a_1b_1u = 0, \\ z(1-z)u_{zz} - xzu_{xz} - yzu_{yz} + tu_{tz} - b_2xu_x - b_2yu_y + \left[c_3 - (a_1+b_2+1)z\right]u_z - a_1b_2u = 0, \\ t(1-t)u_{tt} - xtu_{xt} - ytu_{yt} + zu_{tz} - a_2xu_x - a_2yu_y + \left[c_3 - (a_2+b_1+1)t\right]u_t - a_2b_1u = 0. \end{cases} \tag{11}$$

Here $u(x,y,z,t) := F_{16}^{(4)}(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t)$.

We seek linearly independent solutions as follows:

$$u = x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w, \tag{12}$$

where w is the unknown function and $\alpha, \beta, \gamma, \delta$ are constants to be determined. Introducing new variables $x^{\alpha}y^{\beta}z^{\gamma}t^{\delta}w$ and replacing (11), we obtain the following system of differential equations:

$$\begin{cases} x(1-x)w_{xx}-2xyw_{xy}-xtw_{xt}-xzw_{xz}-y^2w_{yy}-ytw_{yt}-yzw_{yz}-ztw_{zt}+\left[C_1-(A_1+B_1+1)x\right]w_x \\ -(A_1+B_1+1)yw_y-B_1zw_z-A_1tw_t+\left[\alpha(c_1+\alpha-1)x^{-1}-A_1B_1\right]w=0, \\ y(1-y)w_{yy}-x^2w_{xx}-2xyw_{xy}-xzw_{xz}-xtw_{xt}-yzw_{yz}-ytw_{yt}-ztw_{zt}-(A_1+B_1+1)xw_x \\ +\left[C_2-(A_1+B_1+1)y\right]w_y-B_1zw_z+\left[\beta(c_2+\beta-1)y^{-1}-A_1B_1\right]w-A_1tw_t=0, \\ z(1-z)w_{zz}-xzw_{xz}-yzw_{yz}+tw_{zt}+[(A_1+B_2-1)z+C_3]w_z-B_2xw_x-B_2yw_y+\gamma tz^{-1}w_t \\ +[\gamma(\gamma+\delta+c_3-1)z^{-1}-A_1B_2]w=0, \\ t(1-t)w_{tt}-xtw_{xt}-ytw_{yt}+zw_{zt}+\left[C_4-(A_2+B_1+1)t\right]w_t-A_2xw_x-A_2yw_y+\delta zt^{-1}w_z \\ +\left[\delta(c_3+\gamma+\delta-1)t^{-1}-A_2B_1\right]w=0, \end{cases} \tag{13}$$

here

$$\begin{aligned} &A_1=a_1+\alpha+\beta+\gamma,\ A_2=a_2+\delta, \\ &B_1=\alpha+\beta+\delta+b_1,\ B_2=\gamma+b_2, \\ &C_1=c_1+2\alpha,\ C_2=c_2+2\beta,\ C_3=c_3+2\delta+\gamma,\ C_4=c_3+2\gamma+\delta. \end{aligned} \tag{14}$$

System (11) is similar to system (13). Therefore, the following conditions must be met:

$$\begin{cases} \gamma=0, \\ \delta=0, \\ \alpha(c_1+\alpha-1)=0, \\ \beta(c_2+\beta-1)=0, \\ \gamma(c_3+\gamma+\delta-1)=0, \\ \delta(c_3+\gamma+\delta-1)=0. \end{cases} \tag{15}$$

The solutions of system (15) can be written in the form of four combinations:

	1	2	3	4
α	0	1-c_1	0	1-c_1
β	0	0	1-c_2	1-c_2
γ	0	0	0	0
δ	0	0	0	0

(16)

Thus, substituting the solutions (16) of system (15) into equations (12) and performing the necessary simplifications, we obtain linearly independent solutions of the system of second-order partial differential equations (13) in the form of the following four hypergeometric functions:

$$u_1(x,y,z,t)=F_{16}^{(4)}(\alpha+\beta+\gamma+a_1,\delta+a_2,\alpha+\beta+\delta+b_1,\gamma+b_2;2\alpha+c_1,2\beta+c_2,2\gamma+\delta+c_3,2\delta+\gamma+c_3;$$

$$x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(\alpha+\beta+\gamma+a_1)_{m+n+p}(\delta+a_2)_q(\alpha+\beta+\delta+b_1)_{m+n+q}(\gamma+b_2)_p}{(2\alpha+c_1)_m(2\beta+c_2)_n(2\gamma+\delta+c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_2\left(x,y,z,t\right)=x^{1-c_1}F_{16}^{(4)}\left(a_1+1-c_1,a_2,b_1+1-c_1,b_2;2-c_1,c_2,c_3;x,y,z,t\right)$$
$$=x^{1-c_1}\sum_{m,n,p,q=0}^{\infty}\frac{\left(a_1+1-c_1\right)_{m+n+p}\left(a_2\right)_q\left(1-c_1+b_1\right)_{m+n+q}\left(b_2\right)_p}{\left(2-c_1\right)_m\left(c_2\right)_n\left(c_3\right)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_3\left(x,y,z,t\right)=y^{1-c_2}F_{16}^{(4)}\left(a_1+1-c_2,a_2,1-c_2+b_1,b_2;c_1,2-c_2,\ c_3,c_3;x,y,z,t\right)$$
$$=y^{1-c_2}\sum_{m,n,p,q=0}^{\infty}\frac{\left(a_1+1-c_2\right)_{m+n+p}\left(a_2\right)_q\left(1-c_2+b_1\right)_{m+n+q}\left(b_2\right)_p}{\left(c_1\right)_m\left(2-c_2\right)_n\left(c_3\right)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_4\left(x,y,z,t\right)=x^{1-c_1}y^{1-c_2}F_{16}^{(4)}\left(a_1+2-c_1-c_2,a_2,2-c_1-c_2+b_1,b_2;2-c_1,2-c_2,\ c_3,c_3;x,y,z,t\right)$$
$$=x^{1-c_1}y^{1-c_2}\sum_{m,n,p,q=0}^{\infty}\frac{\left(a_1+2-c_1-c_2\right)_{m+n+p}\left(a_2\right)_q\left(2-c_1-c_2+b_1\right)_{m+n+q}\left(b_2\right)_p}{\left(2-c_1\right)_m\left(2-c_2\right)_n\left(c_3\right)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}.$$

Proceeding in the same way as with function $F_{16}^{(4)}$, we can write out solutions for the remaining functions considered in this study: $F_{18}^{(4)}$, $\mathrm{F}_{19}^{(4)}$, $\mathrm{F}_{20}^{(4)}$, $\mathrm{F}_{31}^{(4)}$ functions.

Hypergeometric function (3) $\mathrm{F}_{18}^{(4)}$ that depends on four variables (4) satisfies the following system of equations:

$$\begin{cases}x\left(1-x\right)u_{xx}-2xyu_{xy}-xzu_{xz}+tu_{xt}+\left[c_1-\left(a_1+b_1+1\right)x\right]u_x-y^2u_{yy}-yzu_{yz}\\-\left(a_1+b_1+1\right)yu_y-b_1zu_z-a_1b_1u=0,\\y\left(1-y\right)u_{yy}-2xyu_{xy}-x^2u_{xx}-xzu_{xz}-yzu_{yz}-\left(a_1+b_1+1\right)xu_x+\left[c_2-\left(a_1+b_1+1\right)y\right]u_y\\-b_1zu_z-a_1b_1u=0,\\z\left(1-z\right)u_{zz}-xzu_{xz}-xtu_{xt}-yzu_{yz}-ztu_{zt}+\left[c_3-\left(a_1+b_2+1\right)z\right]u_z\\-a_1tu_t-b_2xu_x-b_2yu_y-a_1b_2u=0,\\t\left(1-t\right)u_{tt}+xu_{xt}-tzu_{tz}+\left[c_1-\left(a_2+b_2+1\right)t\right]u_t-a_2zu_z-a_2b_2u=0.\end{cases}\qquad(17)$$

where is the function $u(x,y,z,t)=F_{18}^{(4)}\left(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t\right)$. The system of differential equations (17) has four linearly independent solutions:

$$u_1\left(x,y,z,t\right)=F_{18}^{(4)}\left(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t\right)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{\left(a_1\right)_{m+n+p}\left(a_2\right)_q\left(b_1\right)_{m+n}\left(b_2\right)_{p+q}}{\left(c_1\right)_{m+q}\left(c_2\right)_n\left(c_3\right)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_2\left(x,y,z,t\right)=y^{1-c_2}F_{18}^{(4)}\left(a_1+1-c_2,a_2,b_1+1-c_2,b_2;c_1,2-c_2,c_3;x,y,z,t\right)$$
$$=y^{1-c_2}\sum_{m,n,p,q=0}^{\infty}\frac{\left(a_1+1-c_2\right)_{m+n+p}\left(a_2\right)_q\left(1-c_2+b_1\right)_{m+n}\left(b_2\right)_{p+q}}{\left(c_1\right)_{m+q}\left(2-c_2\right)_n\left(c_3\right)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_3(x,y,z,t)=z^{1-c_3}F_{18}^{(4)}(a_1+1-c_3,a_2,b_1,b_2+1-c_3;c_1,c_2,2-c_3;x,y,z,t)$$
$$=z^{1-c_3}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1+1-c_3)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2+1-c_3)_{p+q}}{(c_1)_{m+q}(c_2)_n(2-c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_4(x,y,z,t)=y^{1-c_2}z^{1-c_3}F_{18}^{(4)}(a_1+2-c_2-c_3,a_2,1-c_2+b_1,b_2+1-c_3;c_1,2-c_2,2-c_3;x,y,z,t)$$
$$=y^{1-c_2}z^{1-c_3}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1+2-c_2-c_3)_{m+n+p}(a_2)_q(1-c_2+b_1)_{m+n}(1-c_3+b_2)_{p+q}}{(c_1)_{m+q}(2-c_2)_n(2-c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}.$$

The hypergeometric function (4) $F_{19}^{(4)}$ satisfies the following system of equations:

$$\begin{cases}x(1-x)u_{xx}-yyu_{yy}-zxu_{xz}-2xyu_{xy}-yzu_{yz}+\left[c_1-(a_1+b_1+1)x\right]u_x-(a_1+b_1+1)yu_y\\ -b_1zu_z-a_1b_1u=0,\\ y(1-y)u_{yy}-xxu_{xx}-zxu_{xz}-2xyu_{xy}-yzu_{yz}-(a_1+b_1+1)xu_x+\left[c_2-(a_1+b_1+1)y\right]u_y\\ -b_1zu_z-a_1b_1u=0,\\ z(1-z)u_{zz}+t(1-z)u_{zt}-xzu_{xz}-xtu_{xt}-zyu_{yz}-ytu_{yt}-xb_2u_x-yb_2u_y+\left[c_3-(a_1+b_2+1)z\right]u_z\\ -a_1b_2u-a_1tu_t=0,\\ t(1-t)u_{tt}+z(1-t)u_{zt}-a_2zu_z+\left[c_3-(a_2+b_2+1)t\right]u_t-a_2b_2u=0.\end{cases} \tag{18}$$

where the function $u(x,y,z,t)=F_{19}^{(4)}(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t)$. The system of differential equations (18) has following linearly independent solutions:

$$u_1(x,y,z,t)=F_{19}^{(4)}(a_1,a_2,b_1,b_2;c_1,c_2,c_3;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_{p+q}}{(c_1)_m(c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_2(x,y,z,t)=x^{1-c_1}F_{19}^{(4)}(a_1+1-c_1,a_2,b_1+1-c_1,b_2;2-c_1,c_2,c_3;x,y,z,t)$$
$$=x^{1-c_1}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1+1-c_1)_{m+n+p}(a_2)_q(1-c_1+b_1)_{m+n}(b_2)_{p+q}}{(2-c_1)_m(c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_3(x,y,z,t)=y^{1-c_2}F_{19}^{(4)}(a_1+1-c_2,a_2,1-c_2+b_1,b_2;c_1,2-c_2,\ c_3,c_3;x,y,z,t)$$
$$=y^{1-c_2}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1+1-c_2)_{m+n+p}(a_2)_q(1-c_2+b_1)_{m+n}(b_2)_{p+q}}{(c_1)_m(2-c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_4(x,y,z,t)=x^{1-c_1}y^{1-c_2}F_{19}^{(4)}(a_1+2-c_1-c_2,a_2,2-c_1-c_2+b_1,b_2;2-c_1,2-c_2,\ c_3,c_3;x,y,z,t)$$
$$=x^{1-c_1}y^{1-c_2}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1+2-c_1-c_2)_{m+n+p}(a_2)_q(2-c_1-c_2+b_1)_{m+n}(b_2)_{p+q}}{(2-c_1)_m(2-c_2)_n(c_3)_{p+q}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}.$$

The hypergeometric function $F_{20}^{(4)}$ given in (5) satisfies the following system of equations:

$$\begin{cases} x(1-x)u_{xx}-y^2u_{yy}-zxu_{xz}-2xyu_{xy}-yzu_{yz}-(a_1+b_1+1)yu_y+[c_1-(a_1+b_1+1)x]u_x \\ -b_1zu_z-a_1b_1u+tu_{xt}=0, \\ y(1-y)u_{yy}-x^2u_{xx}-zxu_{xz}-2xyu_{xy}-yzu_{yz}-(a_1+b_1+1)xu_x+[c_2-(a_1+b_1+1)y]u_y \\ -b_1zu_z-a_1b_1u=0, \\ z(1-z)u_{zz}-zxu_{xz}-yzu_{yz}+[c_3-(a_1+b_2+1)z]u_z-(xu_x+yu_y+a_1u)b_2=0, \\ (1-t)tu_{tt}+xu_{xt}+[c_1-(a_2+b_3+1)t]u_t-a_2b_3u=0. \end{cases} \tag{19}$$

Here $u(x,y,z,t)=F_{20}^{(4)}(a_1,a_2,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z,t)$. The system of differential equations (19) has four linearly independent solutions of the form:

$$u_1(x,y,z,t)=F_{20}^{(4)}(a_1,a_2,b_1,b_2,b_3;c_1,c_2,c_3;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_2(x,y,z,t)=y^{1-c_2}F_{20}^{(4)}(1-c_2+a_1,a_2,b_1+1-c_2,b_2,b_3;c_1,2-c_2,c_3;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(1-c_2+a_1)_{m+n+p}(a_2)_q(b_1+1-c_2)_{m+n}(b_2)_p(b_3)_q}{(c_1)_{m+q}(2-c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_3(x,y,z,t)=z^{1-c_3}F_{20}^{(4)}(1-c_3+a_1,a_2,b_1,b_2+1-c_3,b_3;c_1,c_2,2-c_3;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(1-c_3+a_1)_{m+n+p}(a_2)_q(b_1)_{m+n}(b_2+1-c_3)_p(b_3)_q}{(c_1)_{m+q}(c_2)_n(2-c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_4(x,y,z,t)=y^{1-c_2}z^{1-c_3}F_{20}^{(4)}(2-c_2-c_3+a_1,a_2,b_1+1-c_2,b_2+1-c_3,b_3;c_1,2-c_2,2-c_3;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(2-c_2-c_3+a_1)_{m+n+p}(a_2)_q(b_1+1-c_2)_{m+n}(b_2+1-c_3)_p(b_3)_q}{(c_1)_{m+q}(2-c_2)_n(2-c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!}.$$

The hypergeometric function $F_{31}^{(4)}$ given in (6) satisfies the following system of equations:

$$\begin{cases} (1-x)u_{xx}-xtu_{xt}-ytu_{yt}+yu_{xy}(1-x)-xu_x(a_1+b_1+1)-a_1tu_t-b_1yu_y+c_1u_x-a_1b_1u=0, \\ y(1-y)u_{yy}+x(1-y)u_{xy}-yu_y(a_1+b_2+1)-b_2xu_x+c_1u_y-a_1b_2u=0, \\ z(1-z)u_{zz}-xzu_{xz}-xtu_{xt}-ztu_{zt}-u_z(a_2z+b_1z+z-c_2)-a_2xu_x-a_2b_1u=0, \\ t(1-t)u_{tt}-ztu_{zt}+c_3tu_t-tu_t(b_3+1+a_2)-b_3zu_z-a_2b_3u=0. \end{cases} \tag{20}$$

Where the function $u(x,y,z,t)=F_{(31)}^{(4)}(a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_3;c_1,c_1,c_2,c_3;x,y,z,t)$. The system of differential equations (20) has four linearly independent solutions of the form:

$$u_1(x,y,z,t)=F_{(31)}^{(4)}(a_1,a_1,a_2,a_2,b_1,b_2,b_1,b_3;c_1,c_1,c_2,c_3;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n)}(a_2)_{p+q}(b_1)_{m+q}(b_2)_p(b_3)_q}{(c_1)_{m+n}(c_2)_p(c_3)_q}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{t^q}{q!},$$

$$u_2(x,y,z,t)=y^{(1-c_2)}F_{31}^{(4)}(a_1+1-c_2,a_1+1-c_2,a_2,a_2,b_1+1-c_2,b_2,b_1+1-c_2,b_3;c_1,c_1,$$
$$2-c_2,c_3;x,y,z,t)=y^{1-c_2}\sum_{m,n,p,q=0}^{\infty}\frac{(a_1+1-c_2)_{m+n}(a_2)_{p+q}(b_1+1-c_2)_{m+q}(b_2)_p(b_3)_q}{(c_1)_{m+n}(2-c_2)_p(c_3)_q}\frac{x^m y^n z^p t^q}{m!n!p!q!},$$

$$u_3(x,y,z,t)=y^{(1-c_3)}F_{31}^{(4)}(a_1,a_1,a_2+1-c_3,a_2+1-c_3,b_1+1-c_3,b_2,b_1+1-c_3,b_3;c_1,c_1,c_2,$$
$$2-c_3;x,y,z,t)=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n}(a_2+1-c_3)_{p+q}(b_1+1-c_3)_{m+q}(b_2)_p(b_3)_q}{(c_1)_{m+n}(c_2)_p(2-c_3)_q}\frac{x^m y^n z^p t^q}{m!n!p!q!},$$

$$u_4(x,y,z,t)=y^{(1-c_2)}t^{(1-c_3)}F_{31}^{(4)}(a_1+1-c_2,a_1+1-c_2,a_2+1-c_3,a_2+1-c_3,$$
$$b_1+2-c_2-c_3,b_2,b_1+2-c_2-c_3,b_3,c_1,c_1,2-c_2,2-c_3;x,y,z,t)$$
$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1+1-c_2)_{m+n}(a_2+1-c_3)_{p+q}(b_1+2-c_2-c_3)_{m+q}(b_2)_p(b_3)_q}{(c_1)_{m+n}(2-c_2)_p(2-c_3)_q}\frac{x^m y^n z^p t^q}{m!n!p!q!}.$$

Discussion

The results of this work demonstrate that hypergeometric functions of four variables have not only theoretical but also applied potential. They can be used to describe multidimensional wave processes, heat conduction, and potential fields. Of particular interest are problems related to the analytical continuation of solutions, determining their regions of convergence, and studying stability under boundary conditions, all of which have direct implications for physical modeling.

However, existing limitations must also be considered. The radius of convergence of general solutions, their asymptotic behavior, and the properties of functions in the vicinity of singular points for real parameter values have not yet been sufficiently studied. Conducting such an analysis will not only expand our understanding of the theory of special functions but also yield new results in the fields of spectral analysis and function spaces.

In the future, an important area of research could be the search for integral representations of the functions under consideration, the study of their orthogonal properties, and the development of numerical methods for constructing approximate solutions. Furthermore, the extension of the obtained results to q-hypergeometric analogs, multicomponent systems, and operator equations appears promising, opening up significant opportunities for further theoretical and applied developments.

Thus, the presented work not only describes new classes of multidimensional hypergeometric functions but also forms the basis for improving analytical and computational methods for their study. The obtained results make a significant contribution to the development of the theory of special functions, spectral analysis, and mathematical physics, deepening the understanding of their role in solving applied problems.

Conclusion

This study investigates the solution of a system of second-order partial differential equations governing four-variable hypergeometric functions, specifically: $F_{16}^{(4)}$, $F_{18}^{(4)}$, $\mathrm{F}_{19}^{(4)}$, $\mathrm{F}_{20}^{(4)}$ and $\mathrm{F}_{31}^{(4)}$. Linearly independent solutions for these systems have been derived, accompanied by an in-depth analysis of their structural and analytical characteristics.

The findings underscore the significance of hypergeometric functions not only in theoretical contexts but also in practical applications. These functions prove essential for analyzing complex multidimensional systems and can be applied in areas such as mathematical physics, engineering, and computational modeling.

By representing these functions as solutions to differential systems, one can achieve a deeper understanding of their solution structures. Moreover, the current results highlight the importance of exploring multidimensional generalizations of hypergeometric functions. Solutions obtained for the four-variable case lay the groundwork for extending the theory to functions of more variables and facilitate the identification of new classes of special functions.

Future research should address the stability of these solutions under various boundary conditions, their asymptotic behavior, and potential integral representations. Such work will enhance the applicability of hypergeometric functions and their derivatives in spatial modeling, spectral analysis, and broader areas of mathematical study.

In summary, this research contributes substantially to the theory of multivariable functions and provides insights into their theoretical and practical relevance.

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