МАТЕМАТИКА ЖӘНЕ ФИЗИКАЛЫҚ ПРОЦЕСТЕР МЕН МЕХАНИКАЛЫҚ ЖҮЙЕЛЕРДІ МАТЕМАТИКАЛЫҚ МОДЕЛЬДЕУ

МАТЕМАТИКА И МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ФИЗИЧЕСКИХ ПРОЦЕССОВ И МЕХАНИЧЕСКИХ СИСТЕМ

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FINITE-DIFFERENCE SCHEME FOR INITIAL-BOUNDARY PROBLEM FOR EQUATION OF DIFFUSION WITH VOLTERRA MEMORY

Abstract

An initial-boundary value problem for a nonlinear diffusion equation with a Volterra-type memory term in time is investigated. The model describes diffusion processes in media with hereditary (long-memory) effects and covers, as particular cases, a range of applications in heat and mass transfer. For its numerical solution we construct an implicit two-level difference scheme with a quadrature approximation of the memory integral. A discrete energy identity is derived, which yields energy stability of the scheme under natural constraints on the time step. The nonlinearity is treated by an inner iterative linearization procedure; at each iteration a tridiagonal system of linear algebraic equations arises and is solved efficiently by the sweep (Thomas) method. The convergence of the scheme is rigorously justified and its convergence rate is estimated in the corresponding energy norm. Numerical experiments are presented that confirm the theoretical error bounds and demonstrate the accurate reproduction of long-memory effects.

Keywords: nonlocality in time; Volterra memory; nonlinear diffusion; implicit difference scheme; sweep method; energy stability; convergence

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ВОЛЬТЕРРА ТИПТІ ЖАДЫ БАР ДИФФУЗИЯ ТЕҢДЕУІ ҮШІН АРАЛАС ЕСЕПКЕ ШЕКТІ АЙЫРЫМДЫҚ СХЕМА

Аңдатпа

Сызықты емес диффузия теңдеуі үшін Вольтерра типті жадылы мүшесі бар бастапқы-шеттік есеп қарастырылады. Мұндай модель тұқым қуалайтын (ұзақ уақыттық жадылық) әсерлері бар ортамен жүретін тасымалдау үдерістерін сипаттап, жылу-массалық алмасу теориясындағы бірқатар қолданбалы есептерді жеке жағдай ретінде қамтиды. Есепті сандық шешу үшін жад интегралын квадратуралық жуықтаумен толықтырылған, уақыт бойынша екі қабатты айқын емес айырымдық схема құрылады. Схема үшін дискреттік энергия теңдігі алынған, соның негізінде уақыт қадамына қойылатын табиғи шектеулер жағдайында оның энергиялық орнықтылығы дәлелденеді. Сызықты емес ішкі итерациялық линеаризация (сызықтандыру) көмегімен ескеріледі; әрбір итерация қадамында үшдиагональды сызықтық алгебралық теңдеулер жүйесі алынады, ол қуалау әдісі арқылы тиімді шешіледі. Ұсынылған схеманың жинақтылығы қатаң түрде негізделген және сәйкес энергетикалық нормада жинақталу реттілігіне баға алынған. Жүргізілген сандық эксперименттер теориялық қателік бағаларын растап, модельде ұзақ уақыттық жадылық әсерлердің дұрыс және физикалық тұрғыдан барабар бейнеленетінін көрсетеді.

Түйін сөздер: уақыт бойынша локалді емес; Вольтерра жады; сызықтық емес диффузия; айқын емес айырымдық схема; қуалау әдісі; энергиялы орнықтылық; жинақтылық.

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ИССЛЕДОВАНИЕ РАЗРЕШИМОСТИ ЧИСЛЕННОГО АНАЛОГА СМЕШАННОЙ ЗАДАЧИ ДЛЯ ПАРАБОЛИЧЕСКОГО УРАВНЕНИЯ С ПАМЯТЬЮ

Аннотация

Рассматривается начально-краевая задача для нелинейного уравнения диффузии с временной памятью типа Вольтерра. Такая модель описывает процессы переноса в средах с наследственными (долговременными) эффектами и охватывает, как частные случаи, ряд задач тепломассопереноса. Для численного решения строится неявная двухслойная разностная схема с квадратурным приближением интеграла памяти. Получено дискретное энергетическое тождество, на основе которого устанавливается энергетическая устойчивость схемы при естественных ограничениях на шаг по времени. Нелинейность учитывается с помощью внутренней итерационной линеаризации; на каждой итерации возникает трёхдиагональная система линейных алгебраических уравнений, эффективно решаемая методом прогонки. Строго доказана сходимость схемы и получена оценка скорости сходимости в соответствующей энергетической норме. Численные эксперименты подтверждают теоретические погрешностные оценки и демонстрируют корректное воспроизведение эффектов долговременной памяти.

Ключевые слова: нелокальность по времени; память Вольтерра; нелинейная диффузия; неявная разностная схема; метод прогонки; энергетическая устойчивость; сходимости.

Introduction

Modeling diffusion and heat conduction in complex media is a central topic in modern physics and engineering. In many practically relevant situations, the current state of a medium depends not only on present conditions but also on its past evolution, a phenomenon usually referred to as memory. Such hereditary effects are observed, for instance, in viscoelastic and polymeric materials, in flow through porous structures, and in thermal systems with delayed response [1-5].

Classical diffusion and heat conduction models, written as parabolic partial differential equations with local-in-time operators, implicitly assume that the medium reacts instantaneously and has no memory. As a consequence, they cannot fully account for hereditary behavior and may lead to incomplete or inaccurate descriptions when memory effects are pronounced [6]. One widely used way to incorporate long-term dependence is to consider parabolic equations with temporal convolution terms, where the memory kernel describes how past states influence the present response [7].

Parabolic equations with memory provide a flexible framework for the description of anomalous diffusion and are capable of reproducing both subdiffusive and superdiffusive dynamics [8-11]. They are particularly useful for modeling fluid transport in porous media and a variety of non-Fickian propagation processes. At the same time, the nonlocal structure of the temporal convolution term introduces additional analytical difficulties and requires more advanced functional-analytic tools when studying well-posedness and stability of solutions, as well as when constructing reliable numerical approximations.

From the numerical standpoint, straightforward explicit schemes for equations with memory often suffer from very restrictive stability conditions and from the necessity to resolve the entire history of the solution. The presence of integral terms in time significantly increases storage requirements and complicates the stability analysis compared with standard parabolic problems. This motivates the development and study of robust implicit or semi-implicit discretizations, for which stability and convergence can be justified using energy methods and a priori estimates [12-14].

There exists a substantial literature on numerical methods for PDEs with memory, including models based on fractional derivatives and Volterra-type integral operators. Various finite-difference and finite-element schemes have been proposed and rigorously analysed for time-fractional diffusion and diffusion-wave equations, with proven stability and convergence properties [12-14]. Similar techniques have also been applied to nonlinear integro-differential equations arising in viscoelasticity and in heat conduction with fading memory kernels [15,16].

In the present paper, we study a mixed initial-boundary value problem for a nonlinear diffusion equation with a Volterra-type memory term. The model combines a nonlinear source with memory effects represented by a temporal convolution with a general kernel. For this problem we construct an implicit finite-difference scheme in which the memory integral is approximated by a discrete convolution, while the nonlinear term is treated iteratively. We derive a discrete energy identity, prove energy stability of the scheme in the ℓ_2-norm under natural constraints on the time step, and obtain convergence estimates consistent with the temporal approximation order. The theoretical results are supported by numerical experiments, which confirm the predicted error behavior and demonstrate the effectiveness of the proposed scheme for modeling diffusion processes with long-time memory.

Methodology of the study

The methodology of the present work is based on a rigorous formulation and analysis of a nonlinear diffusion equation with Volterra-type time memory and on the subsequent construction of a stable finite difference scheme for the corresponding mixed problem.

The problem is considered on the R^2 plane of two independent variables t and x relative to the Cartesian coordinate system. In a half-strip $\Omega \subset R^2$, consider a second-order parabolic equation with memory of the following form:

$$U_t = DU_{xx} + \int_0^t K(x,s)U(x,s)ds + f(U,x,t), \tag{1}$$

where $D > 0$ is the diffusion coefficient, $K(x,t)$ is a memory kernel describing the fading influence of past states, $f(U,x,t)$ represents nonlinear sources or sinks. In this equation (1), the diffusion coefficient describes the intensity of spatial transport, the memory kernel characterises hereditary (long-memory) effects in the medium, and the source term models distributed sources or sinks.

Initial and boundary conditions for the equation (1):

$$U(x,0) = \phi(x), \quad x \in (0,L). \tag{2}$$

$$U(0,t) = g_0(t), \quad U(L,t) = g_L(t), \quad t \in [0,T]. \tag{3}$$

where $\phi(x)$, $g_0(t)$, $g_L(t)$ are smooth functions.

Under these assumptions, we obtain a well-posed mixed initial–boundary value problem for the nonlinear parabolic equation with memory. The goal of the study, formulated in this framework, is to construct an implicit difference scheme for problem (1) - (3) and to justify its energy stability and convergence. All further analytical and numerical steps – derivation of a discrete energy identity, proof of stability, and verification of the convergence order on test problems – are carried out for this mathematically specified model.

Results of the study

In Ω build a difference grid with steps $h=\dfrac{L}{M}$ in spatial and $\tau=\dfrac{T}{N}$ temporal:

The nodal points of the differential grid (indicated by the intersections of the lines $t^n=n\tau,\ n=\overline{0,N};\ x_i=ih,\ i=\overline{0,M}.$) are represented as (x_i,t^n). The numerical solution values at the grid points are represented by.

$$u_i^n\approx U\left(x_i,t^n\right),\ w_i^n\approx K\left(x_i,t^n\right),\ \phi_i=\phi\left(x_i\right),\ g_0^n=g_0\left(t^n\right),\ g_L^n=g_L\left(t^n\right),\ n=\overline{0,N};\ i=\overline{0,M}.$$

To find a numerical solution of the mixed problem (1)-(3) over the difference grid $\Omega_{h\tau}=\left\{\left(x_i,t^n\right)\right\}$, we suggest the following difference splitting scheme in the directions.

We propose the following implicit scheme incorporating memory via discrete convolution:

$$\frac{u_i^{n+1}-u_i^n}{\tau}=D\delta_x^2u_i^{n+1}+\tau\sum_{k=1}^{n}w_i^ku_i^k+\frac{\tau}{2}\left(w_i^0u_i^0+w_i^{n+1}u_i^{n+1}\right)+f\left(u_i^{n+1},x_i,t^{n+1}\right),\tag{4}$$

where: $\delta_x^2u_i^n=\dfrac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2}$ is the discrete Laplacian, w_i^k are weights approximating the memory kernel integral, Nonlinear term is treated implicitly for stability.

Initial (2) and boundary (3) conditions are incorporated at $n=0$ and each time layer accordingly.

$$u_i^0=\phi_i,\ i=0..M.\tag{5}$$

$$u_0^n=g_0^n,\ u_M^n=g_L^n,\ n=0..M\tag{6}$$

Theorem 1. Assume the memory weights $\left\{w_i^k\right\}_k\geq 0$ are nonnegative, nonincreasing, and bounded, and f is Lipschitz. Then the implicit scheme is ℓ^2-stable and satisfies

$$\left\|u^{n+1}\right\|_2^2-\left\|u^n\right\|_2^2+c_0\tau\left\|\nabla_hu^{n+1}\right\|_2^2\leq C\tau\left\|g^{n+1}\right\|_2^2,$$

for all n, with constants independent of τ, h.

Proof. Setting up the notation and inner products. Define the discrete inner product:

$$\left(u,v\right)=h\sum_{i=1}^{M-1}u_iv_i,$$

and norm $\|u\|=\sqrt{\left(u,u\right)}$.

Multiply the scheme by $2\tau u_i^{n+1}$ and sum over i. Multiply both sides of the scheme (4) by $2\tau u_i^{n+1}$, sum from $i=1$ to $M-1$, and write in inner product form:

$$2\tau\left(\frac{u^{n+1}-u^n}{\tau},u^{n+1}\right)=2\tau D\left(\delta_x^2u^{n+1},u^{n+1}\right)+2\tau\left(\tau\sum_{k=1}^{n}w^ku^k+\frac{\tau}{2}\left(w^0u^0+w^{n+1}u^{n+1}\right),u^{n+1}\right)+2\tau\left(f^{n+1},u^{n+1}\right)$$

where $f^{n+1}=(f(u_i^{n+1},x_i,t^{n+1}))$.

Simplify the left-hand side, we get

$$2\tau\left(\frac{u^{n+1}-u^{n}}{\tau},u^{n+1}\right)=2\left(u^{n+1}-u^{n},u^{n+1}\right)=\left\|u^{n+1}\right\|^{2}-\left\|u^{n}\right\|^{2}+\left\|u^{n+1}-u^{n}\right\|^{2} \qquad (7)$$

Estimate the diffusion term. Using discrete Green's formula [17] and boundary conditions (6):

$$\left(\delta_x^2u^{n+1},u^{n+1}\right)=h\sum_{i=1}^{M-1}\frac{\Delta u_i^{n+1}-\Delta u_{i-1}^{n+1}}{h^2}u_i^{n+1}=\frac{1}{h}\sum_{i=0}^{M-1}\Delta u_i^{n+1}\left(u_i^{n+1}-u_{i+1}^{n+1}\right)+\frac{1}{h}\left[u_{M-1}^{n+1}\Delta u_{M-1}^{n+1}-u_0^{n+1}\Delta u_0^{n+1}\right]=$$

$$=-\frac{1}{h}\sum_{i=0}^{M-1}\left(u_{i+1}^{n+1}-u_i^{n+1}\right)^2+\frac{1}{h}\left[u_M^{n+1}\Delta u_{M-1}^{n+1}-\left(\Delta u_{M-1}^{n+1}\right)^2-u_0^{n+1}\Delta u_0^{n+1}\right]\le$$

$$\le -h\sum_{i=0}^{M-1}\left(\frac{u_{i+1}^{n+1}-u_i^{n+1}}{h^2}\right)^2+\frac{1}{2h}\left[\left(u_M^{n+1}\right)^2-\left(\Delta u_{M-1}^{n+1}\right)^2+\left(u_0^{n+1}\right)^2+\left(\Delta u_0^{n+1}\right)^2\right]\le$$

$$\le -\left\|\nabla_h u^{n+1}\right\|^2+\frac{1}{2h}\left[\left(u_M^{n+1}\right)^2+\left(u_0^{n+1}\right)^2\right]=-\left\|\nabla_h u^{n+1}\right\|^2+\frac{1}{2h}\left[\left(g_L^{n+1}\right)^2+\left(g_0^{n+1}\right)^2\right]$$

where the discrete gradient norm is

$$\left\|\nabla_h u^{n+1}\right\|^2=h\sum_{i=0}^{M-1}\left(\frac{u_{i+1}^{n+1}-u_i^{n+1}}{h}\right)^2$$

Thus,

$$2\tau D\left(\delta_x^2u^{n+1},u^{n+1}\right)\le -2\tau D\left\|\nabla_h u^{n+1}\right\|^2+\frac{\tau D}{h}\left[\left(g_L^{n+1}\right)^2+\left(g_0^{n+1}\right)^2\right] \qquad (8)$$

Estimate the memory term. The memory term is

$$2\tau\left(\tau\sum_{k=1}^{n}w^ku^k+\frac{\tau}{2}\left(w^0u^0+w^{n+1}u^{n+1}\right),u^{n+1}\right)=2\tau^2\sum_{k=1}^{n}w^k\left(u^k,u^{n+1}\right)+\tau^2w^0\left(u^0,u^{n+1}\right)+\tau^2w^{n+1}\left\|u^{n+1}\right\|^2$$

by summation by parts (discrete integration by parts).

Using the Cauchy–Schwarz inequality and boundedness of memory weights w_i^n, say $0\le w_i^k\le W$, we get

$$\left|\left(u^k,u^{n+1}\right)\right|\le\left\|u^k\right\|\left\|u^{n+1}\right\|,\ \left|\left(u^0,u^{n+1}\right)\right|\le\left\|u^0\right\|\left\|u^{n+1}\right\|$$

$$2\tau^2\sum_{k=1}^{n}w^k\left(u^k,u^{n+1}\right)+\tau^2w^0\left(u^0,u^{n+1}\right)+\tau^2w^{n+1}\left\|u^{n+1}\right\|^2\le 2\tau^2W\left(\sum_{k=1}^{n}\left\|u^k\right\|+\frac{1}{2}\left\|u^0\right\|\right)\left\|u^{n+1}\right\|+\tau^2W\left\|u^{n+1}\right\|^2$$

We can rewrite this as

$$2\tau^2W\left(\sum_{k=1}^{n}\left\|u^k\right\|+\frac{1}{2}\left\|u^0\right\|\right)\left\|u^{n+1}\right\|\le\varepsilon\left\|u^{n+1}\right\|^2+\frac{1}{\varepsilon}\left[\tau^2W\left(\sum_{k=1}^{n}\left\|u^k\right\|+\frac{1}{2}\left\|u^0\right\|\right)\right]^2$$

$$2\tau^2W\left(\sum_{k=1}^{n}\left\|u^k\right\|+\frac{1}{2}\left\|u^0\right\|\right)\left\|u^{n+1}\right\|+\tau^2W\left\|u^{n+1}\right\|^2\le\left\|u^{n+1}\right\|^2\left(\varepsilon+\tau W\right)+\frac{\tau}{\varepsilon}\left[\tau W\left(\sum_{k=1}^{n}\left\|u^k\right\|+\frac{1}{2}\left\|u^0\right\|\right)\right]^2$$

for any $\varepsilon>0$.

For last sum we can rewrite like this

$$\left(\sum_{k=1}^{n}\left\|u^k\right\|+\frac{1}{2}\left\|u^0\right\|\right)^2=\left(\sum_{k=0}^{n}c_k\left\|u^k\right\|\right)^2,\ \ c_k=\begin{cases}0.5, & k=0\\ 1, & k>0\end{cases}$$

Now we can use Cauchy–Schwarz inequality again

$$\left(\sum_{k=0}^{n}c_k\left\|u^k\right\|\right)^2\le\sum_{k=0}^{n}\left(c_k\right)^2\sum_{k=0}^{n}\left\|u^k\right\|^2=\left(n+\frac{1}{4}\right)\sum_{k=0}^{n}\left\|u^k\right\|^2\le\left(n+1\right)\sum_{k=0}^{n}\left\|u^k\right\|^2$$

Finally,

$$2\tau\left(\tau\sum_{k=1}^{n}w^k u^k+\frac{\tau}{2}\left(w^0u^0+w^{n+1}u^{n+1}\right),u^{n+1}\right)\le\left\|u^{n+1}\right\|^2\left(\varepsilon+\tau W\right)+\frac{W^2\tau^3}{\varepsilon}\left(n+1\right)\sum_{k=0}^{n}\left\|u^k\right\|^2 \quad (9)$$

Estimate the nonlinear term. Assuming $f\left(U,x,t\right)$ satisfies a Lipschitz bound in U, i.e.,

$$\left|f(U,x,t)\right|\le L\left|U\right|+C_f, \quad (10)$$

for constants L,C_f.

Then,

$$\left|2\tau\left(f^{n+1},u^{n+1}\right)\right|\le2\tau\left(L\left\|u^{n+1}\right\|+C_f\right)\left\|u^{n+1}\right\|=2\tau L\left\|u^{n+1}\right\|^2+2\tau C_f\left\|u^{n+1}\right\|,$$

where we applied Young's inequality again.

$$2\tau L\left\|u^{n+1}\right\|^2+2\tau C_f\left\|u^{n+1}\right\|\le2\tau L\left\|u^{n+1}\right\|^2+\tau C_f^{\,2}+\left\|u^{n+1}\right\|^2=\tau\left(2L+1\right)\left\|u^{n+1}\right\|^2+\tau C_f^{\,2} \quad (11)$$

Combine all estimates. Putting all (7) - (9), (11) together:

$$\begin{aligned}\left\|u^{n+1}\right\|^2-\left\|u^n\right\|^2+\left\|u^{n+1}-u^n\right\|^2\le-2\tau D\left\|\nabla_h u^{n+1}\right\|^2+\frac{\tau D}{h}\left[\left(g_M^{n+1}\right)^2+\left(g_0^{n+1}\right)^2\right]-\\+\left\|u^{n+1}\right\|^2\left(\varepsilon+\tau W\right)+\frac{\tau^3W^2\left(n+1\right)}{\varepsilon}\sum_{k=0}^{n}\left\|u^k\right\|^2+\tau\left(2L+1\right)\left\|u^{n+1}\right\|^2+\tau C_f^{\,2}\end{aligned} \quad (12)$$

Notice that $\left\|u^{n+1}-u^n\right\|^2\ge0$ and $-2\tau D\left\|\nabla_h u^{n+1}\right\|^2<0$:

$$\left(\beta-\tau\varepsilon\right)\left\|u^{n+1}\right\|^2-\left\|u^n\right\|^2\le\tau\left[\frac{\tau^2W^2\left(n+1\right)}{\varepsilon}\sum_{k=0}^{n}\left\|u^k\right\|^2+\frac{D}{h}\left[\left(g_M^{n+1}\right)^2+\left(g_0^{n+1}\right)^2\right]+C_f^{\,2}\right]$$

where $\beta=1-\tau\left(2L+1\right)-\tau^2W$.

Choosing that $\varepsilon=\dfrac{\beta}{2\tau}$, we need that $\beta>0\Leftrightarrow\tau\left(2L+1\right)+\tau^2W<1$

$$\frac{\beta}{2}\left\|u^{n+1}\right\|^2-\left\|u^n\right\|^2\le\tau\left[\frac{2\tau^3W^2\left(n+1\right)}{\beta}\sum_{k=0}^{n}\left\|u^k\right\|^2+\frac{D}{h}\left[\left(g_M^{n+1}\right)^2+\left(g_0^{n+1}\right)^2\right]+C_f^{\,2}\right]$$

Simplify $\dfrac{2}{\beta}$

$$\left\|u^{n+1}\right\|^2 \le \frac{2}{\beta}\left\|u^n\right\|^2 + \frac{4\tau^3W^2}{\beta^2}(n+1)\tau\sum_{k=0}^{n}\left\|u_i^k\right\|^2 + \frac{2}{\beta}\tau\frac{D}{h}\left[\left(g_M^{n+1}\right)^2 + \left(g_0^{n+1}\right)^2 + C_f^{\ 2}\right]$$

Use discrete Gronwall with a "triangular" core [18]

$$\left\|u^{n+1}\right\|^2 \le \underbrace{\frac{2}{\beta}}_{a}\left\|u^n\right\|^2 + \underbrace{\frac{4\tau^3W^2}{\beta^2}}_{b}(n+1)\tau\sum_{k=0}^{n}\left\|u_i^k\right\|^2 + \underbrace{\frac{2}{\beta}}_{a}\tau\underbrace{\frac{D}{h}\left[\left(g_L^{n+1}\right)^2 + \left(g_0^{n+1}\right)^2 + C_f^{\ 2}\right]}_{F^{n+1}}$$

When $a \ge 1$, we get

$$\left\|u^{n+1}\right\|^2 \le e^{\gamma t^{n+1} + \delta\left(t^{n+1}\right)^2}\left[\left\|\varphi\right\|^2 + \frac{D}{h}\sum_{j=1}^{n+1}\tau\left[\left(g_L^j\right)^2 + \left(g_0^j\right)^2\right] + t^{n+1}C_f^{\ 2}\right] \tag{13}$$

where $\gamma = \frac{1}{\tau}\ln\frac{2}{\beta}$, $\delta = \frac{4\tau^2W^2}{\beta^2}$, $\beta = 1 - \tau(2L+1) - \tau^2W > 0$ and ϕ - initial (5) condition.

Demonstrating boundedness (stability) of the numerical solution in ℓ_2-norm.

Corollary (energy stability with positive dissipation). Assume the hypotheses of the short ℓ_2*-stability theorem and, in addition,* $W_\infty = \sum_{k\ge 0} w_k < D^2$*. Then there exist* $c_0 = D - \frac{W_\infty}{D} > 0$ *such that for all* $t^n = n\tau \le T$*,*

$$\left\|u^n\right\|^2 + c_0\tau\sum_{j=1}^{n}\left\|\nabla_h u^j\right\|^2 \le e^{\gamma t^n + \delta\left(t^n\right)^2}\left(\left\|\phi\right\|^2 + \frac{D}{h}\sum_{j=1}^{n}\tau\left[\left(g_0^j\right)^2 + \left(g_L^j\right)^2\right] + C_f^2 t^n\right),$$

where $\gamma = \frac{1}{\tau}\ln\frac{2}{\beta}$*,* $\delta = \frac{4\tau^2W^2}{\beta^2}$ *and* $\beta = 1 - \tau(2L+1) - \tau^2W > 0$ *are the constants from the theorem.* If the Dirichlet data (6) vanish, the boundary sum disappears.

Theorem 2. If the assumptions of Theorem 1 hold, then the scheme (4)-(6) converges with order $O\left(\tau + h^2\right)$ *in the* ℓ_2*-norm.*

Proof. Let us introduce consideration $z_i^n = u_i^n - U\left(x_i, t^n\right)$. Substitute $u_i^n = z_i^n + U\left(x_i, t^n\right)$ into (4)-(6) and subtracting the discrete equation satisfied by, we obtain:

$$\frac{z_i^{n+1} - z_i^n}{\tau} = D\delta_x^2 z_i^{n+1} + \frac{\tau}{2}\left(w_i^0 z_i^0 + w_i^{n+1}z_i^{n+1}\right) + \tau\sum_{k=1}^{n} w_i^k z_i^k +$$
$$+\left[f\left(z_i^{n+1} + U_i^{n+1}, x_i, t^{n+1}\right) - f\left(U_i^{n+1}, x_i, t^{n+1}\right)\right] + \Phi_i^{n+1},$$
$$\Phi_i^{n+1} = -\frac{U_i^{n+1} - U_i^n}{\tau} + D\delta_x^2 U_i^{n+1} + \frac{\tau}{2}\left(w_i^0 U_i^0 + w_i^{n+1}U_i^{n+1}\right) + \tau\sum_{k=1}^{n} w_i^k U_i^k + f\left(U_i^{n+1}, x_i, t^{n+1}\right)$$

with homogeneous error data $z_0^n = z_M^n = 0$ and $z_i^0 = 0$. Here Φ_i^{n+1} is local truncation error and $\left\|\Phi^{n+1}\right\| \le C\left(\tau + h^2\right)$.

Taking the h-inner product with z^{n+1} and using the polarization identity, discrete Green's identity for diffusion, Cauchy–Schwarz and Young for the memory term, and the Lipschitz bound for, we obtain the analogue of (12):

$$\left\|z^{n+1}\right\|^2-\left\|z^n\right\|^2+\left\|z^{n+1}-z^n\right\|^2\le-\tau D\left\|\nabla_h z^{n+1}\right\|^2+\frac{\tau W}{D}\sum_{k=0}^{n}\left\|z^k\right\|^2+\tau\left(2L+1\right)\left\|z^{n+1}\right\|^2+\tau\left\|\Phi^{n+1}\right\|^2$$

Using $\left\|\Phi^{n+1}\right\|\le C\left(\tau+h^2\right)$ and Young's inequality gives $\tau\left\|\Phi^{n+1}\right\|^2\le\tau\left\|z^{n+1}\right\|^2+C\tau\left(\tau+h^2\right)^2$ the first part is absorbed on the left. A discrete Gronwall argument (under the step-size condition $\beta=1-\tau\left(2L+1\right)-\tau^2W>0$) yields

$$\left\|z^n\right\|\le C_T\left(\tau+h^2\right),\ 0\le t^n\le T,$$

which proves first-order temporal and second-order spatial convergence in ℓ_2.

Throughout we assume that the source term is Lipschitz in the unknown,

$$\left|f\left(u_1,x,t\right)-f\left(u_2,x,t\right)\right|\le L\left|u_1-u_2\right|,\quad\left(L\ge 0\right)$$

In the present experiment we take

$$K\left(x,t\right)=xt,\ D=10,\ T=1,\ L=1,\ f\left(x,t\right)=2\mathrm{D}t^2-2tx\left(x-1\right)+\frac{t^4x^2\left(x-1\right)}{4}.$$

Which does not depend on u; hence it is Lipschitz with constant $L=0$. This case, the exact solution of the problem:

$$u\left(x,t\right)=t^2x\left(1-x\right)$$

For the trapezoidal memory discretization, the stability estimate yields the condition

$$\tau^2W+\tau\left(2L+1\right)-1<0$$

where $W=\sup\limits_{x\in[0,1],t^n\le T}\sum_{k=0}^{n}\tau w^k\approx\sup\limits_{x\in[0,1],t^n\le T}\int_0^t K\left(x,s\right)ds$. For $K\left(x,s\right)=xs$ one has $\int_0^t K\left(x,s\right)ds=\frac{t^2}{2}x\le\frac{T^2}{2}\Rightarrow W\le\frac{T^2}{2}$. Hence gives the explicit bound:

$$\tau<\tau_{\max}\left(L,W\right)=\begin{cases}\dfrac{1}{2L+1}, & W=0\\ \dfrac{-\left(2L+1\right)+\sqrt{\left(2L+1\right)^2+4W}}{2W}, & W>0\end{cases}$$

With the parameters of our test $T=1\Rightarrow W\le\frac{1}{2}$. Therefore, for the actual nonlinearity here ($L=0$): $\tau<\frac{-1+\sqrt{1+2}}{1}\approx 0.732$. All time steps used below satisfy this constraint.

Below are the results describing two different parts with different strategies for changing parameters, namely, a space step and a time step.

The empirical order of convergence over the time step is determined by the formula

$$P_{k+1}=\frac{\ln\left(e_{k+1}\right)-\ln\left(e_k\right)}{\ln\left(\tau_{k+1}\right)-\ln\left(\tau_k\right)},\ P_{k+1}=\frac{\ln\left(e_{k+1}\right)-\ln\left(e_k\right)}{\ln\left(h_{k+1}\right)-\ln\left(h_k\right)},$$

where e_i is the error in the ℓ_2-norm, τ_k is the time step and h_k is the time step in the k experiment. In the first case, we leave the space step $h = 0.05$ unchanged, changing only the time step by half (Table 1).

Table 1. Temporal refinement with fixed $h = 0.05$.

Experiment number, k	Step in the t variable, τ_k	Step in the x variable, $h = 0.05$	
		Error, e_k	Order, P_{k+1}
1	0.05	$9.3231 \cdot 10^{-5}$	-
2	0.025	$4.6474 \cdot 10^{-5}$	1.00
3	0.0125	$2.3202 \cdot 10^{-5}$	1.00
4	0.00625	$1.1592 \cdot 10^{-5}$	1.00

In the second case, on the contrary, the time step remains constant $\tau = 0.01$, and only the space step changes by half (Table 2).

Table 2. Spatial refinement with fixed $\tau = 0.01$

Experiment number, k	Step in the x variable, h_k	Step in the t variable, $\tau = 0.01$	
		Error, e_k	Order, P_{k+1}
1	0.05	$5.72772 \cdot 10^{-4}$	-
2	0.025	$1.51376 \cdot 10^{-4}$	$1.91(\approx 2.00)$
3	0.0125	$3.841 \cdot 10^{-5}$	$1.98(\approx 2.00)$
4	0.00625	$9.6 \cdot 10^{-6}$	$2.0003(\approx 2.00)$

These data obtained with the help of experimental calculations show (Fig. 1-3) that the numerical solutions of equations (1)-(3) are stable and converge.

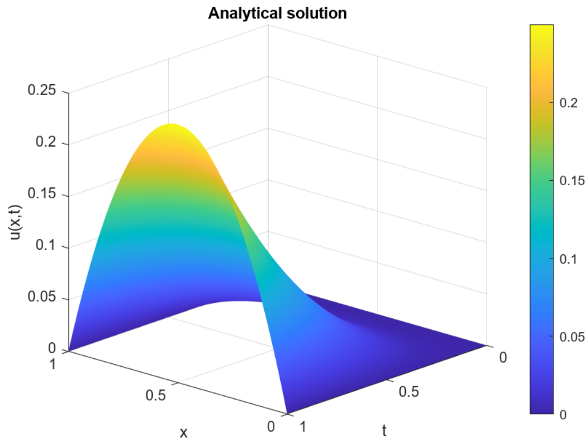

Figure 1. Analytical solution $u(t,x) = t^2 x(1-x)$.

Results in Tables 1-2 demonstrate stability and convergence consistent with theoretical rates. Plots compare numerical and exact solutions at selected times, confirming accuracy.

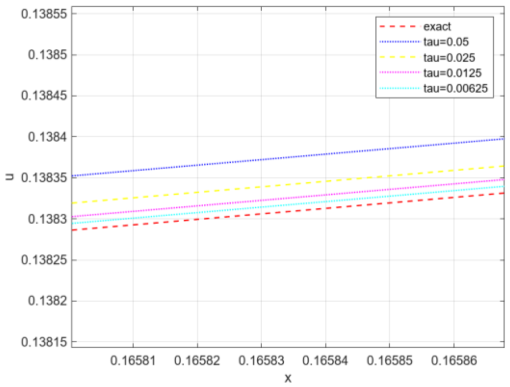

Figure 2. Results of analytical and numerical solution for $h = 0.05$.

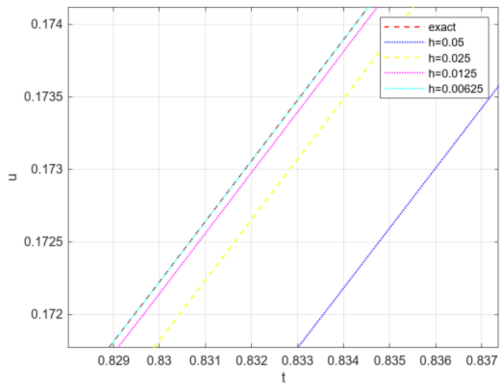

Figure 3. Results of analytical and numerical solution for $\tau = 0.01$

Discussion

The obtained results show that the proposed implicit scheme with Volterra-type memory is a reliable tool for modelling diffusion processes in media with hereditary properties. In contrast to classical schemes for equations without memory and to some explicit methods for equations with integral terms, the constructed algorithm possesses a rigorously proven energy stability and does not require excessively restrictive conditions on the time step. In this respect, it is comparable in theoretical robustness to modern schemes for problems with fractional derivatives and integral memory operators, while preserving a relatively simple implementation (tridiagonal structure and the sweep method). The numerical experiments indicate that the scheme adequately reproduces long-memory effects: the solution is not only stable, but also correctly captures slowed propagation and long-range influence (“tails”) of past states. Prospects for further research include the extension of the approach to multidimensional problems, adaptive time stepping that takes into account the shape of the memory kernel, and the study of a more general class of nonlinearities and possibly degenerate coefficients.

Conclusion

An implicit finite-difference scheme is developed for a nonlinear diffusion equation with Volterra-type memory; a discrete energy identity is derived that guarantees energy stability under natural restrictions on the time step. Convergence of order $O\left(\tau + h^2\right)$ is proved for sufficiently smooth solutions and data; the constants in the estimates are independent of τ and h. The nonlinearity is handled by iterative linearization; at each step a tridiagonal linear system is solved by the Thomas algorithm with linear complexity in the number of grid points. Numerical experiments confirm the claimed orders and stability on tests with smooth and non-smooth data, accurately reproducing long-memory effects. The applicability range covers nonnegative decaying Volterra kernels and standard boundary conditions (Dirichlet/Neumann). Future work includes multidimensional computations, adaptive time stepping, and extending the class of admissible nonlinearities.

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Data Availability Statement

Dataset available on request from the author.

Conflict of Interest

The authors declare that they have no conflicts of interest.

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