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THE DIRICHLET PROBLEM ON THE ORIENTED GRAPHS

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Abstract

Differential operators on graphs often arise in mathematics and different fields of science such as mechanics, physics, organic chemistry, nanotechnology, etc. In this paper the solutions of the Dirichlet problem for a differential operator on a star-shaped graph are deduced. And the differential operator with standard matching conditions in the internal vertices and the Dirichlet boundary conditions at boundary vertices are studied. Task is a model the oscillation of a simple system of several rods with an adjacent end. In work the formula of the Green function of the Dirichlet problem for the second order equation on directed graph is showed. Spectral analysis of differential operators on geometric graphs is the basic mathematical apparatus in solving modern problems of quantum mechanics.

Keywords: oriented graph, vertices of graph, Kifchhoff condition, vibrations of elastic networks, Green's function of Dirichlet problem, solutions of Dirichlet problem.

Аннотация

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ЗАДАЧА ДИРИХЛЕ НА ОРИЕНТИРОВАННОМ ГРАФЕ

Дифференциальные операторы на графах часто возникают в математике и различных областях науки, таких как механика, физика, органическая химия, нанотехнология и т.д. В этой работе были выведены решения задачи Дирихле для дифференциального оператора на графе-звезде. Изучается дифференциальный оператор со стандартными условиями склейки во внутренних вершинах и с граничными условиями Дирихле на граничных вершинах. Так же выведена формула функции Грина задачи Дирихле для уравнения второго порядка на ориентированном графе. Рассматриваемая задача является моделью колебании простой системы из нескольких стержней с примыкающим концом. Спектральный анализ дифференциальных операторов на геометрических графах является основным математическим аппаратом при решении современных проблем квантовой механики.

Ключевые слова: ориентированный граф, вершины графа, условия Кирхгофа, колебания упругих сетей, функция Грина задачи Дирихле, решение задачи Дирихле.

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БАҒЫТТАЛҒАН ГРАФ БОЙЫНДАҒЫ ДИРИХЛЕ ЕСЕБІ

Графтардың бойындағы анықталған дифференциалды операторлар көбінесе математика және механика, физика, органикалық химия, нанотехнология және т.б. сияқты түрлі ғылым салаларында қолданылады. Осы мақалада жұлдыз пішінді граф бойында анықталған дифференциалды оператор үшін Дирихле есебінің шешімдері көрсетіледі. Бұл жұмыста графтың ішкі төбелерінде және Дирихле шарттары, ал шекаралық төбелерінде стандартты жабысу шарттарымен анықталатын дифференциалды операторды қарастырамыз. Сондай-ақ, Дирихле есебінің Грин функциясының формуласы екінші ретті теңдеу үшін көрсетіледі. Қарастырылып отырған есеп бірнеше стерженнен құралған және олар бір төбеде жинақталатын тербелу моделі. Геометриялық графтардың бойныда анықталған дифференциалдық операторлардың спектралдық талдауы кванттық механиканың қазіргі заманғы күрделі есептерін шешудегі негізгі математикалық аппарат болып табылады.

Түйін сөздер: бағытталған граф, графтың төбелері, Кирхгоф шарты, серпімді желілердің тербелістері, Дирихле есебінің Грин функциясы, Дирихле есебінің шешімі.

1. Introduction

In the last 25-30 years, the theory of differential equations and boundary value problems on geometric graphs (spatial networks) has been intensively developed, as evidenced by numerous scientific papers. The beginning of the research was laid in the works (B.C. Pavlov [1], Yu.V. Pokorny, O.M. Penkin ([2], [3]) and others), and foreign (J. von Below ([4], [5]), G. Lumer [6], S. Nicaise [7]) mathematicians and dealt with tasks describing various models: diffusion, vibrations of elastic grids, propagation of nerve impulses, etc. justification of the solvability of boundary value problems on graphs, the study of the structure of the spectrum of these problems, the asymptotics of the spectrum, obtaining estimates for the resolvent. Currently, the most active research is carried out by the creative team of O.M. Penkin [3], B.E.Kanguzhin, L.K.Zhapsarbayeva [8,9].

Let us proceed to the problem formulation. Consider a star-shaped graph G with edges $e_j,\ j=\overline{1,m}$, of equal length π. For each edge e_j introduce the parameter $x_j\in[0,\pi]$. The value $x_j=0$ corresponds to the boundary vertex associated with e_j, and $x_j=\pi$ corresponds to the internal vertex.

The set $W_2^2(G)$ consists functions $y=[y_j]_{j=1}^m$, its components from $y_j\in W_2^2[0,\pi]$. Let H the set of functions y from $W_2^2[0,\pi]$ satisfying the standard matching conditions in the internal vertex

$$\begin{cases} y_1(\pi)=y_j(\pi),\ \ j=2,...,m \\ \sum\limits_{j=1}^{m} y_j'(\pi)=0 \end{cases} \tag{1}$$

In electrical networks, they express the law of Kirchhoff (see [10]), with the vibrations of elastic networks - the balance of voltage, etc. We also give the Dirichlet boundary conditions in the boundary vertices

$$y_j(0)=0,\ \ j=1,...,m \tag{2}$$

On the set of function H we consider the differential equation

$$-y_j''(x_j)+q_j(x_j)y_j(x_j)=\lambda y_j(x_j)+f_j(x_j) \tag{3}$$

where $y=[y_j]_{j=1}^m$ is a vector function on the graph G, λ is the spectral parameter, the so-called potentials $q_j(x_j)$, $j=\overline{1,m}$, are complex-valued functions from $L_2(0,\pi)$, $f_j(x_j)-$ the density distribution of an external force on the edge e_j.

1. The Green's function of the Dirichlet problem

We give some intervening results.

In this subsection we study the question of the existence of the Green's function of the Dirichlet problem

$$-u''(x)+q(x)u(x)=\lambda u(x)+P(x),\ \ 0<x<\pi \tag{4}$$

$$u(0)=0,\ \ u(\pi)=0 \tag{5}$$

By the Green's function we mean a function of two variables $G(x,s,\lambda)$, which is defined when $x\in[0,\pi], s\in[0,\pi]$ and such that for each continuous $P(\cdot)$ on the segment $[0,\pi]$. The solution of the initial boundary value problem (4)-(5) can be represented in the form

$$u(x,\lambda)=\int\limits_0^{\pi} G(x,s,\lambda)P(s)ds.$$

Lemma 1. The solution of the Dirichlet problem (4) - (5) can be represented in the form

$$u(x,\lambda)=\int\limits_0^x \frac{S_0(t,\lambda)S_\pi(x,\lambda)}{D(t,\lambda)}P(t)dt+\int\limits_x^{\pi} \frac{S_\pi(t,\lambda)S_0(x,\lambda)}{D(t,\lambda)}P(t)dt, \tag{6}$$

where $D(t,\lambda)=-S_\pi'(t,\lambda)S_0(t,\lambda)+S_\pi(t,\lambda)S_0'(t,\lambda)$ and the functions $S_0(x,\lambda)$ and $S_\pi(x,\lambda)$ are linearly independent solutions of the homogeneous Cauchy problem

$$-S_0''(x)+q(x)S_0(x)=\lambda S_0(x),\ 0<x<\pi,\ S_0(0,\lambda)=0,\ S_0'(0,\lambda)=1$$

$$-S_\pi''(x)+q(x)S_\pi(x)=\lambda S_\pi(x),\ 0<x<\pi,\ S_\pi(\pi,\lambda)=0,\ S_\pi'(\pi,\lambda)=1$$

Proof. We show that the right-hand side $P(x)$ of (6) is a solution of problem (5).

We first calculate the first derivative

$$u'(x,\lambda)=\int_0^x \frac{S_0(t,\lambda)S'_\pi(x,\lambda)}{D(t,\lambda)}P(t)dt+\int_x^\pi \frac{S_\pi(t,\lambda)S'_0(x,\lambda)}{D(t,\lambda)}P(t)dt+\frac{S_0(\text{x},\lambda)S_\pi(x,\lambda)}{D(\text{x},\lambda)}P(x)-$$
$$-\frac{S_0(\text{x},\lambda)S_\pi(\text{x},\lambda)}{D(\text{x},\lambda)}P(x)$$

hence

$$u'(\text{x},\lambda)=\int_0^x \frac{S_0(t,\lambda)S'_\pi(\text{x},\lambda)}{D(t,\lambda)}P(t)dt+\int_x^\pi \frac{S_\pi(t,\lambda)S'_0(\text{x},\lambda)}{D(t,\lambda)}P(t)dt$$

Then we calculate the second derivative

$$u''(\text{x},\lambda)=\int_0^x \frac{S_0(t,\lambda)S''_\pi(\text{x},\lambda)}{D(t,\lambda)}P(t)dt+\int_x^\pi \frac{S_\pi(t,\lambda)S''_0(\text{x},\lambda)}{D(t,\lambda)}P(t)dt+\frac{S_0(\text{x},\lambda)S'_\pi(\text{x},\lambda)}{D(\text{x},\lambda)}P(x)-\frac{S_\pi(\text{x},\lambda)S'_0(\text{x},\lambda)}{D(\text{x},\lambda)}P(x)$$

or

$$u''(\text{x},\lambda)=\int_0^x \frac{S_0(t,\lambda)S''_\pi(\text{x},\lambda)}{D(t,\lambda)}P(t)dt+\int_x^\pi \frac{S_\pi(t,\lambda)S''_0(\text{x},\lambda)}{D(t,\lambda)}P(t)dt-P(x)$$

Since $S''_0(x,\lambda)=(q(x)-\lambda)S_0(x,\lambda)$, $S''_\pi(x,\lambda)=(q(x)-\lambda)S_\pi(x,\lambda)$, then

$$u''(\text{x},\lambda)=(q(x)-\lambda)\left(\int_0^x \frac{S_0(t,\lambda)S_\pi(\text{x},\lambda)}{D(t,\lambda)}P(t)dt+\int_x^\pi \frac{S_\pi(t,\lambda)S_0(\text{x},\lambda)}{D(t,\lambda)}P(t)dt\right)-P(x)=\left|\overset{from}{(6)}\right|=(q(x)-\lambda)u(\text{x},\lambda)-P(x)$$

this implies the relation (4).

Now let us verify the fulfillment of boundary conditions (5).

We substitute $x=0$ into (6), then obtain

$$u(0,\lambda)=\int_0^\pi \frac{S_\pi(t,\lambda)S_0(0,\lambda)}{D(t,\lambda)}P(t)dt=\left|S_0(0,\overset{as}{\lambda})=0\right|=0$$

Substituting $x=\pi$ into (6), then

$$u(\pi,\lambda)=\int_0^\pi \frac{S_0(t,\lambda)S_\pi(\pi,\lambda)}{D(t,\lambda)}P(t)dt=\left|S_\pi(\pi,\overset{as}{\lambda})=0\right|=0$$

The lemma 1 is proved.

From Lemma 1 the next theorem follows.

Theorem 1. The Green's function of the Dirichlet problem (4) - (5) has the form

$$G_g(x,t,\lambda)=\begin{cases}\dfrac{S_0(t,\lambda)S_\pi(\text{x},\lambda)}{D(t,\lambda)} & when\ 0<t<x\\[2ex] \dfrac{S_\pi(t,\lambda)S_0(\text{x},\lambda)}{D(t,\lambda)} & when\, x<t<\pi\end{cases}$$

where $D(t,\lambda)=-S'_\pi(t,\lambda)S_0(t,\lambda)+S_\pi(t,\lambda)S'_0(t,\lambda)$, $S_0(x,\lambda)$ and $S_\pi(x,\lambda)$ from the lemma 1.

In conclusion, of this subsection, we want to write, that

$$D(\text{t},\lambda)=\begin{vmatrix}S_\pi(\text{t},\lambda) & S_0(\text{t},\lambda)\\ S'_\pi(\text{t},\lambda) & S'_0(\text{t},\lambda)\end{vmatrix}$$

Consequently, the next relation is true $\frac{d}{dt}D(\text{t})=0$. Because of it, we can write

$$D(\mathrm{t},\lambda)=D(0,\lambda)=\begin{vmatrix} S_\pi(0,\lambda) & S_0(0,\lambda) \\ S_\pi^{'}(0,\lambda) & S_0^{'}(0,\lambda)\end{vmatrix}=S_\pi(0,\lambda)=-S_0(\pi,\lambda).$$

2. The Green's function of problem (1) - (2) - (3)

In this section, we calculate the solution $y_1(x_1), y_2(x_2), ..., y_m(x_m), 0<x_j<\pi,\ j=1,...,m$ of problem (1) - (2) - (3) by the right-hand sides $f_1(x_1), f_2(x_2),..., f_m(x_m)$ of equation (3).

First we consider the particular case when $f_2(x_2)=...=f_m(x_m)=0$. That is, by the set of functions $f_1(x_1), f_2(x_2)\equiv 0,..., f_m(x_m)\equiv 0$ must be found $y_1(x_1), y_2(x_2),..., y_m(x_m)$.

Let be e_j - the edge of the graph G. We introduce on the edges e_j the functions $S_{0j}(x_j,\lambda);\ S_{\pi j}(x_j,\lambda)$ that are solutions of the homogeneous Cauchy problem:

$$-y_j^{''}(x_j)+q_j(x_j)y_j(x_j)=\lambda y_j(x_j),$$
$$S_{0j}(0)=0,\ S_{0j}^{'}(0)=1,$$
$$S_{\pi j}(\pi)=0,\ S_{\pi j}^{'}(\pi)=1$$

We introduce the solution of problem (1) - (2) - (3) in the form:

$$\begin{cases} y_2(x_2,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\mathrm{x}_2,\lambda)...S_{0m}(\pi,\lambda) \\ ...\ ...\ ... \\ y_m(x_m,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)...S_{0m}(\mathrm{x}_m,\lambda) \\ y_1(x_1,\lambda)=B_1S_{01}(\mathrm{x}_1,\lambda)S_{02}(\pi,\lambda)...S_{0m}(\pi,\lambda)+\int\limits_0^{\mathrm{x}_1}\frac{S_{01}(t,\lambda)S_{\pi 1}(\mathrm{x}_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt+ \\ +\int\limits_{\mathrm{x}_1}^{\pi}\frac{S_{\pi 1}(t,\lambda)S_{01}(\mathrm{x}_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt \end{cases} \tag{7}$$

where $D_1(t,\lambda)=S'_{01}(t,\lambda)S_{\pi 1}(t,\lambda)-S'_{\pi 1}(t,\lambda)S_{01}(t,\lambda)$

We show that the functions given by system (7) satisfy equations (3), boundary conditions (2), and the relations:

$$y_1(\pi)=y_j(\pi),\ \ j=2,...,m \tag{8}$$

First, let us verify the fulfillment of boundary conditions (2).

Substituting the value $x_1=0,\ x_2=0,\ ...\ ,\ x_m=0$ into (7), we obtain

$$\begin{cases} y_2(0,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(0,\lambda)...S_{0m}(\pi,\lambda)=\left|\overset{\text{так как}}{S_{02}(0,\lambda)=0}\right|=0 \\ ...\ ...\ ... \\ y_m(0,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)...S_{0m}(0,\lambda)=\left|\overset{\text{так как}}{S_{0m}(0,\lambda)=0}\right|=0 \\ y_1(0,\lambda)=B_1S_{01}(0,\lambda)S_{02}(\pi,\lambda)...S_{0m}(\pi,\lambda)+\int\limits_0^{\pi}\frac{S_{\pi 1}(t,\lambda)S_{01}(0,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt=\left|\overset{\text{так как}}{S_{01}(0,\lambda)=0}\right|=0 \end{cases}$$

Let's check conditions (8).

Substituting the value $x_1=\pi,\ x_2=\pi,\ \dots,\ x_m=\pi$ into (7), then

$$\begin{cases} y_2(\pi,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda) \\ \dots\ \dots\ \dots \\ y_m(\pi,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda) \\ y_1(\pi,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda)+\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt=\left|S_{\pi1}\overset{\text{так как}}{(\pi,\lambda)}=0\right|= \\ =B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda) \end{cases} \tag{9}$$

This implies the relation (8).

Now let us verify the fulfillment of equations (3).

We first calculate the first derivative of the relation (7).

$$\begin{cases} y_2'(x_2,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}'(x_2,\lambda)\dots S_{0m}(\pi,\lambda) \\ \dots\ \dots\ \dots \\ y_m'(x_m,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}'(x_m,\lambda) \\ y_1'(x_1,\lambda)=B_1S_{01}'(x_1,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda)+\int\limits_0^{x_1}\frac{S_{01}(t,\lambda)S_{\pi1}'(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,d+\int\limits_{x_1}^{\pi}\frac{S_{\pi1}(t,\lambda)S_{01}'(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt \end{cases}$$

Then we calculate the second derivative of the relation (7).

$$\begin{cases} y_2''(x_2,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}''(x_2,\lambda)\dots S_{0m}(\pi,\lambda) \\ \dots\ \dots\ \dots \\ y_m''(x_m,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}''(x_m,\lambda) \\ y_1''(x_1,\lambda)=B_1S_{01}''(x_1,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda)+\int\limits_0^{x_1}\frac{S_{01}(t,\lambda)S_{\pi1}''(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt+ \\ +\int\limits_{x_1}^{\pi}\frac{S_{\pi1}(t,\lambda)S_{01}''(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt+\frac{S_{01}(x_1,\lambda)S_{\pi1}'(x_1,\lambda)}{D_1(x_1,\lambda)}f_1(x_1)-\frac{S_{\pi1}(x_1,\lambda)S_{01}'(x_1,\lambda)}{D_1(x_1,\lambda)}f_1(x_1)= \\ B_1S_{01}''(x_1,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda)+\int\limits_0^{x_1}\frac{S_{01}(t,\lambda)S_{\pi1}''(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt+\int\limits_{x_1}^{\pi}\frac{S_{\pi1}(t,\lambda)S_{01}''(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt-f_1(x_1) \end{cases}$$

Since $S_{0j}''(x_j,\lambda)=\left(q_j(x_j)-\lambda\right)S_{0j}(x_j,\lambda),\ S_{\pi j}''(x_j,\lambda)=\left(q_j(x_j)-\lambda\right)S_{\pi j}(x_j,\lambda),\ j=1,\dots,m$ then

$$\begin{cases} y_2''(x_2,\lambda)=\left(q_2(x_2)-\lambda\right)B_1S_{01}(\pi,\lambda)S_{02}(x_2,\lambda)\dots S_{0m}(\pi,\lambda)=\left(q_2(x_2)-\lambda\right)y_2(x_2,\lambda) \\ \dots\ \dots\ \dots \\ y_m''(x_m,\lambda)=\left(q_m(x_m)-\lambda\right)B_1S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(x_m,\lambda)=\left(q_m(x_m)-\lambda\right)y_m(x_m,\lambda) \\ y_1''(x_1,\lambda)=\left(q_1(x_1)-\lambda\right)B_1S_{01}(x_1,\lambda)S_{02}(\pi,\lambda)\dots S_{0m}(\pi,\lambda)+ \\ +\left(q_1(x_1)-\lambda\right)\left[\int\limits_0^{x_1}\frac{S_{01}(t,\lambda)S_{\pi1}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt+\int\limits_{x_1}^{\pi}\frac{S_{\pi1}(t,\lambda)S_{01}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt\right]-f_1(x_1)= \\ =\left(q_1(x_1)-\lambda\right)y_1(x_1,\lambda)-f_1(x_1) \end{cases}$$

This implies the relation (3).

Theorem 2. When $f_2(x_2)\equiv 0,...,f_m(x_m)\equiv 0$ the solution of the Dirichlet problem (1) - (2) - (3) can be written in the form

$$\begin{cases} y_1(x_1,\lambda)=-\frac{S_{01}(x_1,\lambda)S_{02}(\pi,\lambda)...S_{0m}(\pi,\lambda)}{\Delta(\lambda)}\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt+\int\limits_0^{x_1}\frac{S_{01}(t,\lambda)S_{\pi1}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)dt+\int\limits_{x_1}^{\pi}\frac{S_{\pi_1}(t,\lambda)S_{01}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)dt \\ y_2(x_2,\lambda)=-\frac{S_{01}(\pi,\lambda)S_{02}(x_2,\lambda)...S_{0m}(\pi,\lambda)}{\Delta(\lambda)}\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt \\ ...\ ...\ ... \\ y_m(x_m,\lambda)=-\frac{S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)...S_{0m}(x_m,\lambda)}{\Delta(\lambda)}\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt \end{cases}$$

where $\Delta(\lambda)=\sum\limits_{j=1}^{m}S_{01}(\pi,\lambda)...S_{0j-1}(\pi,\lambda)S'_{0j}(\pi,\lambda)S_{0j+1}(\pi,\lambda)...S_{0m}(\pi,\lambda)$

Proof. According to the relations (7), the second equality from (1) gets the form

$$B_1\sum_{j=1}^{m}S_{01}(\pi,\lambda)...S_{0j-1}(\pi,\lambda)S'_{0j}(\pi,\lambda)...S_{0m}(\pi,\lambda)=-\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S'_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt \qquad (10)$$

Consider a truncated system

$$\begin{cases} y_2(x_2,\lambda)=B_1S_{01}(\pi,\lambda)S_{02}(x_2,\lambda)...S_{0m}(\pi,\lambda) \\ -\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S'_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt=B_1\sum\limits_{j=1}^{m}S_{01}(\pi,\lambda)...S_{0j-1}(\pi,\lambda)S'_{0j}(\pi,\lambda)...S_{0m}(\pi,\lambda) \end{cases} \qquad (11)$$

This implies

$$y_2(x_2,\lambda)=-\frac{S_{01}(\pi,\lambda)S_{02}(x_2,\lambda)...S_{0m}(\pi,\lambda)}{\Delta(\lambda)}\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S'_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt$$

Similarly, from equations (10) and (7) we obtain

$$y_m(x_m,\lambda)=-\frac{S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)S_{03}(\pi,\lambda)...S_{0m}(x_m,\lambda)}{\Delta(\lambda)}\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S'_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt$$

Consider a truncated system

$$\begin{cases} y_1(x_1,\lambda)=B_1S_{01}(x_1,\lambda)S_{02}(\pi,\lambda)...S_{0m}(\pi,\lambda)+\int\limits_0^{x_1}\frac{S_{01}(t,\lambda)S_{\pi1}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)dt+ \\ +\int\limits_{x_1}^{\pi}\frac{S_{\pi1}(t,\lambda)S_{01}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)dt \\ B_1\Delta(\lambda)=-\int\limits_0^{\pi}\frac{S_{01}(t,\lambda)S'_{\pi_1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)dt \end{cases} \qquad (12)$$

We write the system of solutions in the form

$$\begin{cases} y_1(x_1,\lambda) = -\dfrac{S_{01}(x_1,\lambda)S_{02}(\pi,\lambda)...S_{0m}(\pi,\lambda)}{\Delta(\lambda)}\displaystyle\int_0^{\pi}\dfrac{S_{01}(t,\lambda)S'_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt + \\ +\displaystyle\int_0^{x_1}\dfrac{S_{01}(t,\lambda)S_{\pi1}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt + \displaystyle\int_{x_1}^{\pi}\dfrac{S_{\pi1}(t,\lambda)S_{01}(x_1,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt \\ y_2(x_2,\lambda) = -\dfrac{S_{01}(\pi,\lambda)S_{02}(x_2,\lambda)...S_{0m}(\pi,\lambda)}{\Delta(\lambda)}\displaystyle\int_0^{\pi}\dfrac{S_{01}(t,\lambda)S'_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt \\ \dots\ \dots\ \dots \\ y_m(x_m,\lambda) = -\dfrac{S_{01}(\pi,\lambda)S_{02}(\pi,\lambda)...S_{0m}(x_m,\lambda)}{\Delta(\lambda)}\displaystyle\int_0^{\pi}\dfrac{S_{01}(t,\lambda)S'_{\pi1}(\pi,\lambda)}{D_1(t,\lambda)}f_1(t)\,dt \end{cases} \quad (13)$$

It follows that for $f_2 \equiv f_3 \equiv ... f_k \equiv 0$ and arbitrary $f_1(\cdot)$ the solution of the problem (1) - (2) - (3) is given by the formula (13).

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