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APPLICATION OF THE FICTITIOUS DOMAIN METHOD FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

The article shows the ways of applying the method of fictitious domains in solving problems for ordinary differential equations. In the introduction, a small review of the literature on this method, as well as methods for the numerical solution of these problems, is made. The problem statement for the method of fictitious domains for ordinary differential equations is considered. Further, the inequality of estimates was shown. The solution of the auxiliary problem approximates the solution of the original problem with a certain accuracy. The inequality of estimates is obtained in the class of generalized solutions. For the purpose of visual application of the fictitious domain method in problems, a boundary value problem for a one-dimensional nonlinear ordinary differential equation is considered. The problem was written in the form of a difference scheme and led to a solution using the sweep method. In the numerical solution of the problem, numerical calculations were carried out for various values of the parameter included in the auxiliary problem, based on the method of fictitious domains. The numbers of iterations, execution time, and graphs of these calculations are presented and analyzed.

Keywords: fictitious domain method, solution estimation, sweep method, second-order nonlinear differential equation, numerical calculations.

Аңдатпа

ЖАЙ ДИФФЕРЕНЦИАЛДЫҚ ТЕҢДЕУЛЕР ҮШІН ЖАЛҒАН АЙМАҚТАР ӘДІСІНІҢ ҚОЛДАНУЫ

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Мақалада жай дифференциалдық теңдеулер үшін жалған аймақтар әдісін есептерді шығаруда қолдану жолдары көрсетілген. Кіріспеде осы әдіс бойынша әдебиеттерге, сондай-ақ осы мәселелерді сандық шешу әдістеріне шағын шолу жасалды. Жай дифференциалдық теңдеулер үшін жалған аймақтар үшін есептің қойылымы қарастырылды. Ары қарай жалған аймақтар әдісі бойынша көмекші есептің қойылымын толық келтіре отырып, көмекші есеп шешімі бастапқы есеп шешімін белгілі бір дәлдікпен жуықтайтынын бағалау теңсіздігі арқылы көрсетілді. Бағалау теңсіздігі жалпыланған шешімдер класында алынды. Жалған аймақ әдісінің есептерде нақты қолдануы көрсету мақсатында бірөлшемді сызықты емес жай дифференциалдық теңдеуі үшін шекаралық есебі қарастырылды. Есепті айырымдық сұлба түрінде жазылып, қуалау әдісі бойынша шешуге келтірілді. Есепті сандық шешу барасында жалған аймақтар әдісіне негізделген көмекші есепке кіретін параметрдің әртүрлі мәндері үшін сандық есептеулер жүргізілді. Сол есептеулердің итерация саны, орындалу уақыты және графиктері келтіріліп салыстырылды.

Түйін сөздер: жалған аймақ әдісі, шешімді бағалау, қуалау әдісі, екінші ретті сызықты емес дифференциалдық теңдеу, сандық есептеулер.

Аннотация

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ПРИМЕНЕНИЕ МЕТОДА ФИКТИВНЫХ ОБЛАСТЕЙ ДЛЯ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

В статье показаны способы применения метода фиктивных областей при решении задач для обыкновенных дифференциальных уравнений. В введении сделан небольшой обзор литературы по данному методу, а также методам численного решения этих задач. Рассмотрено постановка задачи для метода фиктивных областей для обыкновенных дифференциальных уравнений. Далее было показано неравенство оценок решение вспомогательной задачи с определенной точностью приближает решение исходной задачи. Неравенство оценок получено в классе обобщенных решений. С целью наглядного применения метода фиктивных области в задачах рассмотрена граничная задача для одномерного нелинейного обыкновенного дифференциального уравнения. Задачу записали в виде разностной схемы и привели к решению по методу прогонки. При численном решении задачи были проведены численные расчеты для различных значений параметра, входящего в вспомогательную задачу, на основе метода фиктивных областей. Приведены и анализированы числа итераций, время выполнения и графики этих вычислений.

Ключевые слова: метод фиктивных области, оценка решения, метод прогонки, нелинейное дифференциальное уравнение второго порядка, численные расчеты.

1 Introduction. The method of fictitious domain are widely used for numerical solution of problems of mathematical physics in a free zone [1,2]. With A.N. Konovalov [2], he developed the theory of the fictitious areas to solve the problem of free zones in the quantitative implementation of discrete models. On its basis, a new class of local-two-way approximations was created in direct and spectral problems.

R. Glowinski, T-W. Pan, J. In the work of Periaux [3], a set of fictitious domain methods is considered, based on the actual use of the Lagrange multiplier, defined at a specific boundary. Due to true boundary conditions, the proposed method is often used to simulate viscous non-compressible potential flows. According to the proposed methodology, the initial differential problem has an effective control problem with the tribune point, and the iterative method of combined gradients is used for numerical implementation.

2 Report presentation

$$Lu \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u = f(x), \tag{1}$$

$$x = (x_1, x_2, ..., x_n) \in D_1,$$

$$u(x) = 0, \quad x \in \partial D_1. \tag{2}$$

(1), (2) let's assume that the solution and coefficients of the problem are sufficiently smooth $a_{ij}(x) = a_{ji}(x), \quad x(c) \geq 0$ and let next elliptic conditions satisfied

$$\inf_{x\in D_1} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i \xi_j \geq \mu \sum_{i=1}^{n} \xi_i^2 \tag{3}$$

μ (1), (2) in the method of fictitious domain to solve the first limit problem (FDM) ξ does not depend on a positive constant

$$D = D_1 \cup D_2,$$

$$D_1 \cup D_2 = S.$$

$$L_\varepsilon \upsilon \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x) \frac{\partial \upsilon}{\partial x_j} \right) + c\upsilon = F(x) \tag{4}$$

where

$$A_{ij}(x)=\begin{cases} a_{ij}(x), & x\in D_1,\\ 0,\ x\in D_2,\ i\neq j, \\ \varepsilon^{-2},\ x\in D_2,\ i=j. \end{cases}$$

$$c(x)=\begin{cases} c(x), & x\in D_1,\\ 0, & x\in D_2. \end{cases}$$

$$F(x)=\begin{cases} f(x), & x\in D_1,\\ 0, & x\in D_2. \end{cases}$$

(4) we set the limit conditions for the equation

$$\upsilon(x)=0,\quad x\in\partial D. \tag{5}$$

$$\left[\upsilon(x)\right]|_S=0,\quad \left[\sum_{i,j=1}^{n} A_{ij}(x)\cos(\nu,x_i)\frac{\partial\upsilon}{\partial x_j}\right]|_S=0. \tag{6}$$

where ν - S normal to the boundary; $[\cdot]|_S$ sign is S function jump on the surface. Let's estimate the difference $\omega(x)=u(x)-\upsilon(x)$

3 False areas method solution evaluation.

$$L_\varepsilon\omega(x)=0,\ x\in D,\quad x\notin S, \tag{7}$$

$$\omega(x)=0,\ x\in\partial D, \tag{8}$$

$$\left[\omega(x)\right]|_S=0,\quad \left[\sum_{i,j=1}^{n} A_{ij}(x)\cos(\nu,x_i)\frac{\partial\omega}{\partial x_i}\right]|_S=\varphi(x), \tag{9}$$

where $\varphi(x)=\sum_{i,j=1}^{n} a_{ij}\cos(\nu,x_i)\frac{\partial u}{\partial x_i},\quad x\in S$.

(7) equations $\omega(x)$ and multiply (8) and (9) subject to conditions D We obtain the following equilibrium by integrating

$$\int_{D_1}\left(\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial\omega}{\partial x_i}\frac{\partial\omega}{\partial x_j}+c\omega^2\right)dx+\frac{1}{\varepsilon^2}\int_{D_2}\sum_{i=1}^{n}\left(\frac{\partial\omega}{\partial x_i}\right)^2dx=-\int_S\varphi(x)\omega(x)ds. \tag{10}$$

We remove the first term of the left part of formula (10) and apply the Cauchy-Schwartz inequality to the right, then

$$\frac{1}{\varepsilon^2}\int_{D_2}\sum_{i=1}^{n}\left(\frac{\partial\omega}{\partial x_i}\right)^2dx\le\sqrt{\int_S\varphi^2(x)ds}\sqrt{\int_S\omega^2(x)ds} \tag{11}$$

we take.

Width to assessment the positive side $0<\delta<\delta_0$ was ω_δ We use the inequality true (11) for the boundary band

$$\int_S\omega^2(x)ds\le C_1\left(\delta\int_{\omega_\delta}\omega^2(x)dx+\frac{1}{\delta}\int_{\omega_\delta}\sum_{i=1}^{n}\left(\frac{\partial\omega}{\partial x_i}\right)^2dx\right) \tag{12}$$

and D equal to zero in the boundary part of the region, $\omega(x)$ which was true for the Friedrichs inequality

$$\int_D \omega^2(x)dx \le C_2 \int_D \sum_{i=1}^n \left(\frac{\partial \omega}{\partial x_i}\right)^2 dx \tag{13}$$

we use. (12) where $\delta = \delta_0$ selecting and continuing to integrate to the right of the inequality, we obtain the following

$$\int_S \omega^2(x)dS \le C_3 \left(\int_{D_2} \omega^2(x)dx + \int_{D_2} \sum_{i=1}^n \left(\frac{\partial \omega}{\partial x_i}\right)^2 dx \right) \tag{14}$$

$\omega(x) = 0$ $x \in \partial D$ since (14)'s the Friedrichs inequality can be applied to the first connector on the right (13). From here we take

$$\int_S \omega^2(x)dS \le C_4 \int_{D_2} \sum_{i=1}^n \left(\frac{\partial \omega}{\partial x_i}\right)^2 dx \tag{15}$$

(15) Taking into account the inequality, we obtain the following estimate from (11):

$$\left(\int_{D_2} \sum_{i=1}^n \left(\frac{\partial \omega}{\partial x_i}\right)^2 dx \right)^{1/2} \le C_5 \varepsilon^2 \tag{16}$$

and use (13) we take

$$\left(\int_{D_2} \omega^2(x)dx \right)^{1/2} \le C_6 \varepsilon^2 . \tag{17}$$

Similarly, we obtain the inequality from equation (10) using the elliptical condition (3)

$$\mu \int_{D_1} \sum_{i=1}^n \left(\frac{\partial \omega}{\partial x_i}\right)^2 dx \le \sqrt{\int_S \varphi^2 dS} \sqrt{\int_S \omega^2(x)dS} \le C_7 \varepsilon^2 . \tag{18}$$

Using Friedrich's generalized inequality

$$\int_{D_1} \omega^2(x)dx \le C_8 \left(\int_S \omega^2(x)dS + \int_{D_1} \sum_{i=1}^n \left(\frac{\partial \omega}{\partial x_i}\right)^2 dx \right). \tag{19}$$

and (15), (16) and (18) from the estimates leads to the following inequality

$$\int_{D_2} \omega^2(x)dx \le C_8 \varepsilon^2 . \tag{20}$$

Thus, we have the problem (4) - (6) $\upsilon(x)$ solution of the problem (1) - (2) $u(x)$ decision $W_2^1(D_1)$, ε We have proved that the approximation is accurate, ie

$$\|u - \upsilon\|_{W_2^1(D_1)} \le C_9 \varepsilon \tag{21}$$

where C_8 constant does not depend of ε .

Free limited area D_1 The following scheme of approximate solution of the first boundary value problem can be given (1), (2). D_1 the smallest area D cover with a parallelepiped. (4)-(6) so that the solution of the problem approximates the solution of the original problem with the required accuracy we choose ε . Then we solve the problem (4) - (6) by the difference method with the required accuracy.

4 Numerical solution of the problem.

One-dimensional statement of the problem:

$$y'' - 2yy' = f(x) \quad 0 < x < 0.5 \tag{22}$$

$$y(0) = y(0.5) = 0 \tag{23}$$

where

$$f(x) = \frac{(3-2x)\cdot e^{2x} - 1}{1-e} - \frac{e^{4x} - e^{2x}}{(1-e)^2} + 2x \tag{24}$$

The exact solution of the problem (22), (23):

$$y(x) = \frac{1 - e^{2x}}{2(e-1)} + x \tag{25}$$

Auxiliary report for the fictitious domain method:

$$\frac{d}{dx}\left(a(x)\frac{dv}{dx}\right) - 2v\frac{d}{dx}(b(x)v) = f^{\varepsilon}(x) \quad 0 < x < 1 \tag{26}$$

$$v(0) = v(1) = 0 \tag{27}$$

$$[v]|_{x=0.5} = \left[a(x)\frac{dv}{dx} - b(x)v\right]|_{x=0.5} = 0 \tag{28}$$

where

$$a(x) = \begin{cases} 1, & 0 < x < 0.5 \\ \frac{1}{\varepsilon^2}, & 0.5 < x < 1 \end{cases}$$

$$b(x) = \begin{cases} 1, & 0 < x < 0.5 \\ \frac{1}{\varepsilon}, & 0.5 < x < 1 \end{cases}$$

$$f^{\varepsilon}(x) = \begin{cases} f(x), & 0 < x < 0.5 \\ 0, & 0.5 < x < 1 \end{cases}$$

Inheritance method for nonlinear second order differential equations. To construct the difference form of equation (26), we reduce it to the following form:

$$\frac{dv}{dt} = \frac{d}{dx}\left(a(x)\frac{dv}{dx}\right) - 2v\frac{d}{dx}(b(x)v) - f^{\varepsilon}(x)$$

We write differently:

$$\frac{v_i^{n+1} - v_i^n}{\tau} = \frac{1}{h}\left[a_{i+1/2}\frac{v_{i+1}^{n+1} - v_i^{n+1}}{h} - a_{i-1/2}\frac{v_i^{n+1} - v_{i-1}^{n+1}}{h}\right] -$$

$$-\frac{(v_i^n - |v_i^n|)\cdot\left(b_{i+1}v_{i+1}^{n+1} - b_i v_i^{n+1}\right) + (v_i^n + |v_i^n|)\cdot\left(b_i v_i^{n+1} - b_{i-1}v_{i-1}^{n+1}\right)}{h} - f_i^{\varepsilon}$$

where

$$a_{i+1/2} = \frac{a_{i+1} + a_i}{2}, \qquad a_{i-1/2} = \frac{a_i + a_{i-1}}{2}$$

We write the last equation in a convenient form

$$A_i \cdot v_{i-1}^{n+1} - B_i \cdot v_i^{n+1} + C_i \cdot v_{i+1}^{n+1} = D_i \tag{29}$$

where

$$A_i = \tau\left(\frac{a_{i-1/2}}{h^2} + \frac{(v_i^n + |v_i^n|)}{h} b_{i-1}\right),$$

$$B_i = \left(1 + \tau\frac{a_{i+1/2} + a_{i-1/2}}{h^2} + 2\tau\frac{|v_i^n|}{h} b_i\right)$$

$$C_i = \tau\left(\frac{a_{i+1/2}}{h^2} - \frac{(v_i^n - |v_i^n|)}{h} b_{i+1}\right)$$

$$D_i = -(v_i^n - \tau f_i^\varepsilon)$$

We solve equation (29) by inheritance.

Let's do numerical calculations.[0; 1] segment of the grid with a step $h = 1/N$ Let's build a grid in steps, where $N = 100$. Iteration $\|v^{n+1} - v^n\| < \epsilon = 10^{-4}$ stop under the condition.Numerical calculations$\varepsilon = \{1; \quad 0.1; \quad 0.01\}$ was conducted on three grounds.

Table 1. The results of the calculation by the method of inheritance

ε values	*n number of iterations*	*Execution time (in seconds)*
1	*88*	*4.27*
0.1	*28*	*1.31*
0.01	*16*	*0.51*

In order to test the algorithm for numerical solution of the problem, we first solve the problem in the segment and compare the graphs of the numerical solution with the exact analytical solution (Figure 1).

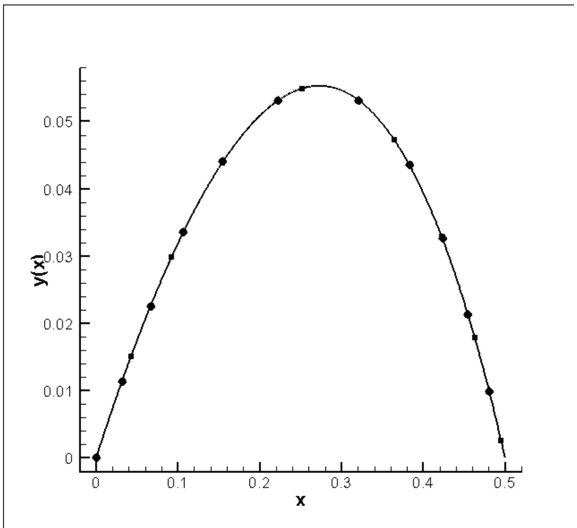

Figure 1. $y(x)$'s function graph
■ approximate solution;● exact solution

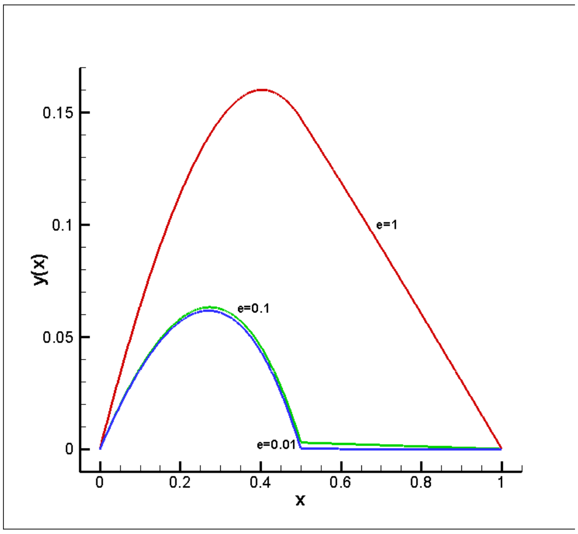

Figure 2. ε for a different value $v(x)$ function graph

5 Conclusion.

We see that the method of inheritance of our chosen numerical solution corresponds to the exact solution. Therefore, you can see that the algorithm is chosen correctly. Looking at the second figure, we can see from the graph that different values of the parameter obtained for the method of false zones have a significant impact on the solution of the problem. We also notice that the parameter value has an effect on the number and time of iterations of the report. We see that the method of inheritance of our chosen numerical solution corresponds to the exact solution. Therefore, you can see that the algorithm is chosen correctly. Looking at the second figure, we can see from the graph that different values of the parameter obtained for the method of false zones have a significant impact on the solution of the problem. We also notice that the parameter value has an effect on the number and time of iterations of the report.

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References

1 Вабищевич П.Н. Метод фиктивных областей для задачи математической физики. – М.:Изд-во МГУ, 1991.-156 с.

2 Коновалов А.Н. Задачи фильтрации многофазной несжимаемой жидкости. 3-е изд. Singapure et al.: World Scientific, 1994.173 с.

3 R. Glowinski , T-W. Pan , J. Periaux, A fictitious domain method for Dirichlet problem and applications. Computer Methods in Applied Mechanics and Enginering, 1994, 111, pp.283-303.

4 Шокин Ю.И., Данаев Н.Т., Хакимзянов Г.С., Шокина Н.Ю. Лекции по разностным схемам на подвижных сетках – Алматы, 2008. Ч.2. – 184 с.

5 Eiseman P.R. Adaptive grid generation // Comput. Meth. Appl. Mech. Engng. 1987. V. 64. P. 321 - 376.

6 133. Hawken D.F., Gottlieb J.J., Hansen J.S. Review article. Review of some adaptive node - movement techniques in finite - element and finite-difference solutions of partial differential equations // J. Comput. Phys. 1991. V. 95. № 2. P. 254 - 302.

7 Thompson J.F. Grid generation techniques in computational dynamics // AlAA Journal. 1984. V. 22. P. 1505 - 1523.

8 Thompson J.F., Warsi Z.U.A., Mastin C.W. Numerical grid generation, foundations and applications. – 1985, New York, etc.: Elsevier.

9 Лисейкин В.Д. Методы построения разностных сеток: Моногр. / Новосиб.гос.ун-т. Новосибирск, 2014. 208 с.

10 Прокопов Г.П. об организации сравнения алгоритмов и программ построения регулярных двумерных разностных сеток. М. 1989. (Препринт № 18. АН СССР. ИПМ им. Келдыша. 27 с.)

11 Garanzha V.A. Computation of discrete curvatures based on polar polyhedra theory. Proceedings of International Conference "Numerical geometry, grid generation and scientific computing", Moscow, 10-13 June 2008, M.: Folium, 2008. P.182-189.

12 Garanzha V.A. Approximation of the curvature of Alexandrov surfaces using dual polyhedra // Rus. J. Numer. Analys. Modeling. 2009. V.24. 5. P.409-423.

13 Garanzha V.A. Discrete extrinsic curvatures and approximation of surfaces by polar polyhedra // Ж. вычисл. матем. и матем. физ. 2010. Т.50. 1. С.71-98.

14 Z. Xie, R. Sevilla, O. Hassan, K. Morgen, The generation of arbitrary order curved meshes for 3D finite element analysis. Computational Mechanics, 51(3),361-374, 2013.

15 J.-F. Remacle, J. Lambrechts, C. Geuzaine, and T. Toulorge, Optimizing the geometrical accuracy of 2D curvilinear meshes. Procedia Engineering, 82, 228–239, 2014

16 Temirbekov, N., Malgazhdarov, Y., Tokanova, S., Baigereyev, D., Turarov, A. Information technology for numerical simulation of convective flows of a viscous incompressible fluid in curvilinear multiply connected domains / Journal of Theoretical and Applied Information Technology, 2019, 97(22), pp. 3166–3177

References

1 Vabishhevich P.N. (1991) Metod fiktivnyh oblastej dlja zadachi matematicheskoj fiziki[The method of fictitious domains for a mathematical physics problem]. – M.:Izd-vo MGU, p.156.(in Russian)

2 Konovalov A.N. (1994) Zadachi fil'tracii mnogofaznoj neszhimaemoj zhidkosti [Problems of filtration of a multiphase incompressible fluid]. 3-e izd. Singapure et al.: World Scientific. p.173. (in Russian)

3 R. Glowinski , T-W. Pan, J. Periaux, A fictitious domain method for Dirichlet problem and applications. Computer Methods in Applied Mechanics and Enginering, 1994, 111, pp.283-303.

4 Shokin Ju.I., Danaev N.T., Hakimzjanov G.S., Shokina N.Ju. (2008) Lekcii po raznostnym shemam na podvizhnyh setkah[Lectures on difference schemes on movable grids]. Almaty, part 2. p.184. (in Russian)

5 Eiseman P.R. Adaptive grid generation // Comput. Meth. Appl. Mech. Engng. 1987. V. 64. P. 321 - 376.

6 Hawken D.F., Gottlieb J.J., Hansen J.S. Review article. Review of some adaptive node - movement techniques in finite - element and finite-difference solutions of partial differential equations // J. Comput. Phys. 1991. V. 95. № 2. P. 254 - 302.

17 Thompson J.F. Grid generation techniques in computational dynamics // AlAA Journal. 1984. V. 22. P. 1505 - 1523.

7 Thompson J.F., Warsi Z.U.A., Mastin C.W. Numerical grid generation, foundations and applications. – 1985, New York, etc.: Elsevier.

8 Lisejkin V.D. (2014) Metody postroenija raznostnyh setok [Methods for constructing difference grids]: Monogr. / NSU. Novosibirsk. p.208. (in Russian)

9 Prokopov G.P. (1989) Ob organizacii sravnenija algoritmov i programm postroenija reguljarnyh dvumernyh raznostnyh setok [On the organization of comparison of algorithms and programs for constructing regular two-dimensional difference grids]. M. (Preprint № 18. AN SSSR. IPM im. Keldysha. 27 p.) (in Russian)

10 Garanzha V.A. Computation of discrete curvatures based on polar polyhedra theory. Proceedings of International Conference "Numerical geometry, grid generation and scientific computing", Moscow, 10-13 June 2008, M.: Folium, 2008. P.182-189.

11 Garanzha V.A. Approximation of the curvature of Alexandrov surfaces using dual polyhedra // Rus. J. Numer. Analys. Modeling. 2009. V.24. 5. P.409-423.

12 Garanzha V.A. (2010) Discrete extrinsic curvatures and approximation of surfaces by polar polyhedral.Zh. Vychisl. mat. and mat. physical T.50. 1,.71-98.

13 Z. Xie, R. Sevilla, O. Hassan, K. Morgen, The generation of arbitrary order curved meshes for 3D finite element analysis. Computational Mechanics, 51(3), 361-374, 2013.

14 J.-F. Remacle, J. Lambrechts, C. Geuzaine, and T. Toulorge, Optimizing the geometrical accuracy of 2D curvilinear meshes. Procedia Engineering, 82, 228–239, 2014

15 Temirbekov, N., Malgazhdarov, Y., Tokanova, S., Baigereyev, D., Turarov, A. Information technology for numerical simulation of convective flows of a viscous incompressible fluid in curvilinear multiply connected domains / Journal of Theoretical and Applied Information Technology, 2019, 97(22), pp. 3166–3177