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ANALYSIS OF A PARABOLIC APPROXIMATION FOR A LINEAR UNSTEADY NAVIER–STOKES PROBLEM

Abstract

The paper investigates a parabolic approximation of a linear unsteady Navier–Stokes problem for an incompressible viscous fluid. The proposed approach replaces the incompressibility constraint with an evolutionary pressure equation that includes a small regularization parameter. Such a transformation makes it possible to apply methods from the theory of parabolic differential equations and simplify the analysis of the mathematical model. The study employs techniques of functional analysis, energy estimates, and the Galerkin method to examine the properties of the approximating system. Conditions ensuring the existence, uniqueness, and regularity of generalized and strong solutions are established. Particular attention is paid to the convergence of the approximating solution to the solution of the original Navier–Stokes system as the approximation parameter tends to zero. Quantitative estimates characterizing the convergence rate in various functional spaces are obtained. To demonstrate the practical relevance of the proposed approach, a model problem related to transient flow in a pipeline system is considered. The results indicate that the parabolic approximation provides a satisfactory balance between computational efficiency and solution accuracy. The estimates obtained justify the method's applicability to the mathematical modeling of unsteady hydrodynamic processes and may serve as a basis for developing efficient numerical algorithms in computational fluid dynamics.

Keywords: Navier–Stokes equations; parabolic approximation; incompressible fluid; unsteady flow; convergence analysis; energy estimates; computational fluid dynamics.

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СЫЗЫҚТЫҚ СТАЦИОНАР ЕМЕС НАВЬЕ–СТОКС ЕСЕБІНІҢ ПАРАБОЛАЛЫҚ ЖУЫҚТАУЫН ЗЕРТТЕУ

Аңдатпа

Мақалада тұтқыр сығылмайтын сұйықтықтың қозғалысын сипаттайтын сызықтық стационар емес Навье-Стокс есебінің параболалық жуықтау тәсілі қарастырылады. Ұсынылған модельде сығылмайтындық шарты қысым үшін енгізілген эволюциялық теңдеумен алмастырылады, бұл бастапқы есепті зерттеуді жеңілдетуге және параболалық типтегі теңдеулер теориясының әдістерін қолдануға мүмкіндік береді. Зерттеу барысында функционалдық талдау, энергетикалық бағалау әдістері және Галеркин тәсілі пайдаланылды. Аппроксимацияланған жүйе үшін шешімнің бар болуы мен жалғыздығы дәлелденіп, оның тұрақтылығы мен тегістігіне қатысты қасиеттері зерттелді. Сонымен қатар, жуықталған шешімнің бастапқы Навье–Стокс жүйесінің шешіміне жақындау заңдылықтары қарастырылып, жуықтау параметрінің әсерін сипаттайтын сандық бағалаулар алынды. Алынған нәтижелер параболалық жуықтаудың математикалық тұрғыдан негізделген тәсіл екенін көрсетеді. Жүргізілген модельдік есептеу тәжірибесі параметрдің кішіреюімен қателіктің азаятынын және теориялық қорытындылардың сандық нәтижелермен сәйкес келетінін растады. Ұсынылған әдіс гидродинамикалық процестерді модельдеу кезінде қолданылатын тиімді есептеу алгоритмдерін әзірлеуге негіз бола алады.

Түйін сөздер: Навье-Стокс теңдеулері; параболалық жуықтау; сығылмайтын сұйықтық; тұрақсыз ағын; шешімдердің конвергенциясы; энергия бағалаулары; есептеу сұйықтық динамикасы.

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ИССЛЕДОВАНИЕ ПАРАБОЛИЧЕСКОЙ АППРОКСИМАЦИИ ЛИНЕЙНОЙ НЕСТАЦИОНАРНОЙ ЗАДАЧИ НАВЬЕ-СТОКСА

Аннотация

В статье рассматривается параболическая аппроксимация линейной нестационарной задачи Навье–Стокса для вязкой несжимаемой жидкости. Исследуемый подход основан на замене условия несжимаемости специальным эволюционным уравнением для давления, содержащим малый параметр. Такая модификация позволяет привести исходную систему к более удобной для анализа форме и использовать методы теории параболических дифференциальных уравнений при исследовании свойств решения. Для изучения аппроксимирующей модели применяются методы функционального анализа, энергетические оценки и метод Галеркина. Получены условия существования и единственности обобщенного решения, а также установлены требования, при которых решение обладает повышенной регулярностью. Особое внимание уделено исследованию сходимости аппроксимирующей системы к исходной задаче. Получены количественные оценки погрешности, характеризующие зависимость точности решения от параметра аппроксимации и позволяющие определить скорость сходимости в соответствующих функциональных пространствах. Практическая значимость работы связана с возможностью использования предложенного подхода при моделировании нестационарных гидродинамических процессов. Рассмотренный пример течения жидкости в трубопроводной системе демонстрирует, что параболическая аппроксимация обеспечивает приемлемое сочетание вычислительной эффективности и точности расчётов. Полученные результаты могут быть использованы при разработке численных методов решения задач гидродинамики и математической физики.

Ключевые слова: уравнения Навье-Стокса; параболическая аппроксимация; несжимаемая жидкость; нестационарное течение; сходимость решений; энергетические оценки; вычислительная гидродинамика.

Main Provisions

This paper investigates a parabolic approximation of the linear nonstationary Navier–Stokes problem for an incompressible fluid. This approximation is based on replacing the incompressibility condition with an evolution equation for pressure with a small parameter. For the approximating system under consideration, a priori estimates are obtained that ensure the existence and uniqueness of a generalized solution, and conditions are established under which the solution is more regular. An analysis of the convergence of the approximation to the solution of the original Navier–Stokes problem as ε tends to zero is presented. Quantitative error estimates are constructed in various functional spaces, allowing one to determine the rate of convergence of the approximate solution. It is shown that decreasing the approximation parameter leads to a consistent reduction in error and ensures that the results are consistent with the original hydrodynamic model. The results confirm the mathematical correctness of the approach under consideration and provide a theoretical basis for developing effective computational algorithms to model nonstationary processes in hydrodynamics and mathematical physics.

Introduction

Non-stationary hydrodynamic problems occupy an important place in mathematical physics and applied modeling. The Navier–Stokes equations, which underlie most modern hydrodynamic models, are traditionally used to describe the motion of a viscous, incompressible fluid [1–3]. Despite the significant amount of research devoted to this system, the issues of constructing effective approximate models and substantiating their accuracy remain relevant. This is due both to the complexity of the analytical study of the original system and the need to develop computationally efficient methods for solving practical problems. One of the main difficulties in analyzing non-stationary Navier-Stokes problems is the presence of an incompressibility condition, which determines the relationship between the velocity field and pressure. Therefore, approaches that regularize the original system and replace the incompressibility constraint with an additional evolution equation for pressure are of considerable interest. One such approach is the parabolic approximation method, which allows one to reduce the original problem to a more convenient form for analysis and utilize the apparatus of the theory of parabolic equations [4-7].

The practical significance of such methods is confirmed by their relevance to computational fluid dynamics, which is used to model flows in pipeline systems, hydraulic structures, ventilation systems, and other engineering objects. As computational demands increase, models that provide an acceptable balance between solution accuracy and computational costs are particularly important.

This paper considers the linear non-stationary Navier-Stokes problem and its corresponding parabolic approximation. The focus is on investigating the mathematical properties of the approximating system. Theorem 1 establishes the conditions for the existence, uniqueness, and regularity of a solution to the problem under consideration. Theorem 2 then derives estimates for the convergence of the approximating solution to the solution of the original system as the approximation parameter tends to zero. The estimates obtained enable assessment of the accuracy of the proposed approximation and analysis of its applicability to nonstationary fluid-flow problems. In addition, the results may be useful in developing numerical algorithms for hydrodynamic simulations.

Research Methodology

The methodological basis of the study is based on mathematical analysis, the theory of partial differential equations, and the apparatus of function spaces. This paper examines the linear nonstationary Navier-Stokes problem for an incompressible fluid and its parabolic approximation, which contains a small regularization parameter. The study aims to establish the mathematical correctness of the approximating model and determine the nature of its convergence to the original system. In the first stage, the original boundary value problem is formulated, and the corresponding parabolic approximation is constructed, in which the incompressibility condition is replaced by an evolution equation for the pressure. This approach allows us to consider the resulting system within the framework of parabolic equations theory and to utilize known results on the solvability of the corresponding initial-boundary value problems [7].

The choice of parabolic regularization is based on its ability to preserve the fundamental properties of the original problem while simplifying the mathematical analysis. Unlike artificial compressibility methods, which introduce an additional pseudo-time parameter and necessitate studying limit transitions across several parameters, the approximation under consideration yields a parabolic system, allowing the direct application of classical results in the theory of parabolic equations. Compared to penalty methods, which require selecting a large penalty coefficient and subsequently monitoring its influence on solution stability, the approach used provides a more transparent interpretation of the regularization parameter and enables quantitative error estimates. Unlike projection methods, which are primarily focused on numerical problem solving, parabolic approximation enables an analytical study of the existence, regularity, and convergence of solutions. These features determine the choice of this method as the basis for this study.

To prove the existence and uniqueness of a solution, the Galerkin method is used, based on constructing finite-dimensional approximations and subsequent limiting. A priori estimates are obtained using the energy method, which involves multiplying equations by special test functions and integrating over the spatial domain and time. To substantiate the obtained estimates, Hölder and Young inequalities, as well as embedding and trace theorems for functions in Sobolev spaces, are used [8]. The convergence study is based on comparing the solutions of the original and approximating problems. For this purpose, a representation of the difference in solutions through auxiliary functions is introduced, for which a special boundary value problem is constructed. Energy inequalities are then derived, allowing error estimates to be obtained in various functional norms. Analysis of the dependence of these estimates on a small parameter enables us to establish the order of convergence of the approximation and to justify its use in the study of non-stationary hydrodynamic processes. Thus, the methodology combines analytical methods from the theory of differential equations and functional analysis, providing a rigorous mathematical justification for the parabolic approximation under consideration.

Statement of the Problem

Let us consider in the space-time domain

$$Q_T = \Omega \times (0,T), \qquad \Omega \subset \mathbb{R}^n, \qquad n = 2,3$$

a linear non-stationary Navier-Stokes problem for an incompressible fluid

Equation of motion

$$\vec{v}_t + \nabla p = \Delta\vec{v} + \vec{f}, \tag{1}$$

Incompressibility condition

$$div\ \vec{v} = 0, \tag{2}$$

Initial and boundary conditions

$$\vec{v}|_{t=0} = \vec{v}^0(x),\ \vec{v}|_{\partial\Omega\times(0,T)} = 0, \tag{3}$$

Here $\vec{v}(x,t)$ is the fluid velocity vector, $p(x,t)$ is the pressure, $\vec{f}(x,t)$ is the external mass force, and $\vec{v}^0(x)$ is the initial velocity distribution. Condition (2) expresses the incompressibility of the medium, and the boundary condition in (3) corresponds to the no-slip condition of the fluid at the boundary of the domain.

To construct a parabolic approximation of system (1)-(3), we replace the incompressibility constraint (2) with an evolution equation for pressure. This yields the following approximation problem:

$$\vec{v}_t^\varepsilon + \nabla p^\varepsilon = \Delta\vec{v}^\varepsilon + \vec{f} - \nabla Q_m, \tag{4}$$

$$div\ \vec{v}^\varepsilon = \varepsilon(\Delta p^\varepsilon - p_t^\varepsilon), \tag{5}$$

$$\vec{v}|_{t=0} = \vec{v}^0(x),\ \vec{v}|_{\partial\Omega\times(0,T)} = 0,\ p^\varepsilon|_{t=0} = p_0(x),\ \left.\frac{\partial p^\varepsilon}{\partial n}\right|_{\partial Q_T} = 0. \tag{6}$$

Here $p_0(x)$ is the value of the solution $p(x,t)$ of problem (1)–(3) at the initial moment $t = 0$. It is known to be determined from the Neumann problem

$$\Delta p_0 = div\ \vec{f}\big|_{t=0},$$

$$\left.\frac{\partial p_0}{\partial n}\right|_{\partial\Omega} = [\Delta\vec{v}^0 \cdot n]|_{\partial\Omega} + \left[\vec{f}\cdot n\right]\big|_{\partial\Omega}. \tag{7}$$

The function $Q_m(x),\ m = 1,2\ ...$ is present in equation (4) to satisfy the m-th order matching condition

$$\left.\frac{\partial^k}{\partial t^k}(p^\varepsilon - Q_m)\right|_{t=0} = \left.\frac{\partial^k p}{\partial t^k}\right|_{t=0}, \qquad k = 1,2,..,m.$$

Its differential properties are described in terms of the derivatives $D_t^m\vec{f},\ D_t^m(\vec{v})|_{t=0}$ или $D_x^{2m}(\vec{v}(x))$. Since we will be describing a solution with a zero-order time derivative, for simplicity we will set $Q_m(x)=0$.

Zero-order matching condition

$$p^\varepsilon|_{t=0}=p|_{t=0}$$

Is implemented as the initial condition in the pressure problem.

We now turn to the study of the properties of the solution to problem (4)–(6). First, we obtain a priori estimates that ensure the existence and uniqueness of a generalized solution and allow us to establish its regularity under additional assumptions on the initial data. The corresponding results are formulated in the following theorem.

Theorem 1. A) Let $\vec{f}(x,t)\in L_2\big(0,T;\ W_2^{-1}(\Omega)\big), \vec{v}^0(x)\in L_2(\Omega)$. Then problem (4)-(6) has a unique generalized solution $\vec{v}^\varepsilon(x,t)\in L_\infty(0,T;\ L_2(\Omega))\cap L_2\big(0,T;\ W_2^1(\Omega)\big)$ and the following estimate holds

$$\max_{0\le t\le T}\|\vec{v}^\varepsilon\|_{L_2(\Omega)}^2+\int_0^T\|\nabla\vec{v}^\varepsilon\|_{L_2(\Omega)}^2\,dt\le C\left(\int_0^t\|\vec{f}\|_{W_2^{-1}(\Omega)}^2\,dt+\|\vec{v}^0\|_{L_2(\Omega)}^2\right) \tag{8}$$

B) Let $\vec{f}(x,t)\in L_2(Q_T),\ \vec{v}^0(x)\in W_2^1(\Omega),\ \partial Q_T\in C^2$. Then the generalized solution of problem (4)-(6) becomes strong, i.e. $\vec{v}^\varepsilon(x,t)\in W_2^{1,2}(Q_T),\ \nabla p^\varepsilon\in L_2(Q_T)$ and the estimate holds

$$\|\vec{v}^\varepsilon\|_{W_2^{1,2}(Q_T)}+\|\nabla p^\varepsilon\|_{L_2(Q_T)}\le C_1\cdot\varepsilon^{-1/2} \tag{9}$$

The proof is carried out by the Galerkin method [8] based on estimates (8), (9). The first of them is obtained by multiplying equation (4) by $\vec{v}^\varepsilon$ в $L_2(\Omega)$, and the second is a consequence of estimates of the solution of boundary value problems for parabolic equations and the estimate

$$\|\nabla p^\varepsilon\|_{L_2(\mathrm{Q_T})}\le C_2\cdot\varepsilon^{-\frac{1}{2}},$$

which is obtained simultaneously with (8).

After establishing the existence, uniqueness, and regularity of the solution to problems (4)–(6), it is natural to study the convergence of the constructed approximation to the solution of the original system (1)–(3). To do this, we obtain a priori estimates of the difference between the corresponding solutions and determine the order of their dependence on the small parameter ε. The results are formulated in the following theorem.

Theorem 2. Let $\vec{f}_t(x,t),\ \nabla\vec{f}(x,t)\in W_2^{-1}\big(0,T;\ L_2(\Omega)\big), \vec{v}^0(x)\in W_2^5(\Omega),\ \partial\Omega\subset C^3$. Then the following estimates take place:

$$\int_0^T\|\vec{v}^\varepsilon-\vec{v}\|_{W_2^1(\Omega)}^2dt+\max_{0\le t\le T}\|\vec{v}^\varepsilon-\vec{v}\|_{L_2(\Omega)}^2\le C_3\cdot\varepsilon^{\frac{3}{2}}, \tag{10}$$

$$\|\vec{v}^\varepsilon-\vec{v}\|_{W_2^{2,1}(\mathrm{Q_T})}\le C_4\cdot\varepsilon^{1/2} \tag{11}$$

Proof. First of all, we note that for the given $\vec{f},\vec{v}^0$ the functions $p_{tt}(x,t),\ \Delta p_t(x,t)\in L_2(Q_T)$. Next, we assume that the representation is valid

$$\vec{v}^\varepsilon=\vec{v}+\varepsilon\vec{\omega},\ \ p^\varepsilon=p+\varepsilon\pi.$$

Then the functions $\vec{\omega},\pi$ are the solution of the boundary value problem

$$\omega_t + \nabla\pi = \Delta\omega, \tag{12}$$

$$div\,\vec{\omega} = \Delta p^{\varepsilon} - p_t^{\varepsilon}, \tag{13}$$

$$\vec{\omega}|_{t=0} = 0, \tag{14}$$

Let's get an auxiliary estimate for π. To do this, we integrate (12) with respect to t.

$$\vec{\omega}(t) + \int_0^t \nabla\pi dt = \nu \int_0^t \Delta\vec{\omega}dt.$$

Fair enough

$$\left\| \int_0^t \pi dt \right\|_{L_2(\Omega)} \le C \left\| \nabla \int_0^t \pi dt \right\|_{W_2^1(\Omega)} \le$$
$$\le C\left(\max_{0\le t\le T} \|\vec{\omega}(t)\|_{L_2(\Omega)} + \nu \left[\int_0^t \|\nabla\vec{\omega}\|_{L_2(\Omega)} dt \right]^{\frac{1}{2}} \right) = C \cdot V(t) \tag{15}$$

Now we multiply equation (12) by $\vec{\omega}$ in $L_2(\Omega)$ and take into account (13), (14)

$$\frac{1}{2}\frac{d}{dt}\|\vec{\omega}\|^2_{L_2(\Omega)} + \nu\|\nabla\vec{\omega}\|^2_{L_2(\Omega)} + \frac{\varepsilon}{2}\frac{d}{dt}\|\pi\|^2_{L_2(\Omega)} + \varepsilon\|\nabla\pi\|^2_{L_2(\Omega)} =$$

$$= -\int_\Omega \pi(p_t - \Delta p)dx - \int_{\partial\Omega} \pi \frac{\partial p}{\partial n} d(\partial\Omega).$$

Let us integrate the resulting equality with respect to t.

$$\frac{1}{2}\|\vec{\omega}(t)\|^2_{L_2(\Omega)} + \nu\int_0^t \|\nabla\vec{\omega}\|^2_{L_2(\Omega)}dt + \frac{\varepsilon}{2}\|\pi(t)\|^2_{L_2(\Omega)} + \varepsilon\int_0^t \|\nabla\pi\|^2_{L_2(\Omega)}dt \le$$
$$\le -\int_0^t \left[\int_\Omega \pi(p_t - \Delta p)dx + \int_{\partial\Omega} \pi\frac{\partial p}{\partial n}d(\partial\Omega) \right] d\tau \tag{16}$$

We write the second term on the right-hand side as follows:

$$\int_0^t \int_{\partial\Omega} \pi \frac{\partial p}{\partial n} d(\partial\Omega)\mathrm{d}\tau = \int_0^t \int_{\partial\Omega} \frac{\partial}{\partial\tau}\int_0^\tau \pi d\xi \frac{\partial p}{\partial n} d\tau d(\partial\Omega).$$

Integrating by parts, we find

$$\int_0^t \int_{\partial\Omega} \pi\frac{\partial p}{\partial n}d(\partial\Omega)\mathrm{d}\tau = \int_{\partial\Omega} \left(\int_0^t \pi d\xi \right) \frac{\partial p}{\partial n}(t) d(\partial\Omega) - \int_0^t \int_{\partial\Omega} \left(\int_0^\tau \pi d\xi \right) \frac{\partial p_\tau}{\partial n} d(\partial\Omega) d\tau.$$

The first term on the right-hand side is estimated similarly to the second one using the trace theorem and Young's inequality, so it is absorbed by the left-hand side of the energy inequality for a sufficiently small choice of the parameter δ_1.

From here

$$\int_0^t \int_{\partial\Omega} \pi \frac{\partial p}{\partial n} d(\partial\Omega) \mathrm{d}\tau = \int_0^t \left\| \int_0^\tau \pi d\xi \right\|_{L_2(\partial\Omega)} \left\| \frac{\partial p_\tau}{\partial n} \right\|_{L_2(\partial\Omega)} d\tau.$$

The inequalities are fair

$$\left\| \int_0^\tau \pi d\xi \right\|_{L_2(\partial\Omega)} \leq C \left\| \int_0^\tau \nabla\pi d\xi \right\|_{L_2(\Omega)}^{\frac{1}{2}} \left\| \int_0^\tau \pi d\xi \right\|_{L_2(\Omega)}^{\frac{1}{2}},$$

$$\left\| \frac{\partial p_\tau}{\partial n} \right\|_{L_2(\partial\Omega)} \leq C \|p_\tau\|_{W_2^2(\Omega)}$$

That's why

$$\int_0^t \int_{\partial\Omega} \pi \frac{\partial p}{\partial n} d(\partial\Omega) \mathrm{d}\tau \leq C \int_0^t \left\| \int_0^\tau \nabla\pi d\xi \right\|_{L_2(\Omega)}^{\frac{1}{2}} \left\| \int_0^\tau \pi d\xi \right\|_{L_2(\Omega)}^{\frac{1}{2}} \|p_\tau\|_{W_2^2(\Omega)} d\tau \leq$$

$$\leq \int_0^t \left[\varepsilon \frac{\delta_1}{4} \left\| \int_0^\tau \nabla\pi d\xi \right\|_{L_2(\Omega)}^2 + \frac{\delta_1}{4} \left\| \int_0^\tau \pi d\xi \right\|_{L_2(\Omega)}^2 + \frac{\delta_1^{-1}}{2} C^2 \|p_\tau\|_{W_2^2(\Omega)}^2 \cdot \varepsilon^{-1/2} \right] d\tau$$

Using the Hölder inequalities and (15), we conclude

$$\int_0^t \int_{\partial\Omega} \pi \frac{\partial p}{\partial n} d(\partial\Omega) \mathrm{d}\tau \leq \frac{T\varepsilon\delta_1}{4} \int_0^t \|\nabla\pi\|_{L_2(\Omega)}^2 d\tau + \frac{T\delta_1}{4} V^2(t) + \frac{\delta_1^{-1}}{2} C^2 \varepsilon^{-1/2} \int_0^t \|p_\tau\|_{W_2^2(\Omega)}^2 d\tau$$

The first integral in (16) is estimated similarly

$$\int_0^t \int_\Omega \pi(\Delta p - p_t) dx \leq \frac{T\delta_2}{2} V^2(t) + \frac{\delta_2^{-1}}{2} C \int_0^t \left[\|\Delta p_t\|_{L_2(\Omega)}^2 + \|p_{tt}\|_{L_2(\Omega)}^2 \right] d\tau.$$

Thus, for sufficiently small δ_1, δ_2 we conclude

$$\begin{gathered} \frac{1}{2} \max_{0 \leq t \leq T} \|\vec{\omega}\|_{L_2(\Omega)}^2 + \frac{\varepsilon}{2} \max_{0 \leq t \leq T} \|\pi\|_{L_2(\Omega)}^2 + \nu \int_0^T \|\nabla\vec{\omega}\|_{L_2(\Omega)}^2 dt + \varepsilon \int_0^T \|\nabla\pi\|_{L_2(\Omega)}^2 dt \leq \\ \leq C \left[\left(\varepsilon^{-\frac{1}{2}} + 1 \right) \int_0^T \|\Delta p_t\|_{L_2(\Omega)}^2 dt + \int_0^T \|p_{tt}\|_{L_2(\Omega)}^2 dt \right] \end{gathered} \tag{17}$$

Estimate (17) implies inequalities (10) and (11). The theorem is proven.

The theoretical results obtained in Theorems 1 and 2 provide the basis for using the approximation under consideration in computational fluid dynamics problems. The existence, regularity, and convergence estimates allow the approximating system to be considered a mathematically sound alternative to the original model for numerical calculations. The practical value of this approach is most evident in problems that require multiple solutions to non-stationary fluid flow equations, where reducing computational complexity is particularly important. Let us consider a possible example of the practical application of the results obtained.

Example of Practical Application

The relevance of developing effective methods for modeling hydrodynamic processes is underscored by the growing use of computational fluid dynamics (CFD) in industry. According to analytical forecasts, the global CFD market is showing steady positive growth, reflecting increasing demand for mathematical models and computational algorithms that deliver high computational accuracy at reasonable computational cost [9].

Hydrodynamic modeling plays a special role in the operation of pipeline systems, the transportation of liquids and gases, and the design of energy infrastructure facilities. The development of trunk pipeline networks and the need to monitor transient equipment operation require methods that enable rapid evaluation of flow parameters and prediction of changes in hydrodynamic characteristics over time [10].

Under these conditions, parabolic approximation methods are of practical interest for their ability to reduce the computational complexity of modeling while maintaining acceptable accuracy.

As an example of practical application, consider modeling unsteady oil flow in a main pipeline after a change in a pumping station's operating mode. When pumping equipment starts or stops, transient processes occur in the system, accompanied by changes in flow velocity and pressure distribution along the pipeline. To ensure safe operation, it is necessary to quickly predict flow parameters and assess the likelihood of undesirable hydrodynamic phenomena, including pressure surges and hydraulic shocks.

The mathematical description of such processes is based on the equations of motion of a viscous incompressible fluid. In practical simulations, repeated solution of the complete Navier–Stokes equations can be computationally expensive, especially when transient regimes must be analyzed repeatedly.

The estimates obtained in this study show that for small values of the approximation parameter, the solution to the approximate problem converges to the solution of the original system with a known order of accuracy. This enables the proposed approach to be used for operational forecasting of flow parameters in pipeline networks, as well as for developing software systems to monitor and manage oil and gas infrastructure. These estimates enable the proposed approach to be used for operational forecasting of flow parameters in pipeline networks and for developing monitoring systems for oil and gas infrastructure. To test the viability of the proposed approach and quantify its accuracy, a computational model experiment is also conducted. Despite the use of a simplified computational domain, the results obtained allow us to assess the convergence of the approximation and draw conclusions about its applicability to modeling real hydrodynamic processes.

Numerical Experiment

To illustrate the theoretical results, a computational model experiment was conducted. The test problem was flow in the square domain $\Omega = [0, 1] \times [0, 1]$. The reference solution was defined by smooth analytical functions of velocity and pressure satisfying the initial and boundary conditions of the problem under consideration. The approximate solution was computed for various values of the parameter ε, and the accuracy was assessed using the L2-norm relative errors of the velocity and pressure. The results presented in Table 1 demonstrate a steady decrease in the approximate solution error as the approximation parameter ε decreases. The most significant decrease in errors is observed when moving from $\varepsilon = 10^{-1}$ to $\varepsilon = 10^{-2}$, after which the nature of the change becomes smoother. In particular, a decrease in ε from 10^{-1} to 10^{-4} leads to a decrease in the relative velocity error from 1.738% to 0.055%, and the pressure error from 10.285% to 0.325%. The obtained data are consistent with theoretical convergence estimates and confirm that a decrease in the ε parameter ensures a consistent convergence of the solutions of the approximating and original problems.

Table 1. Quantitative assessment of the accuracy of the parabolic approximation for various values of the parameter ε

Approximation parameter ε	*Relative velocity error, %*	*Relative pressure error, %*	*Calculation time, c*
10^{-1}	1,738	10,285	13,35
10^{-2}	0,55	3,252	14,7
10^{-3}	0,174	1,028	16,05
10^{-4}	0,055	0,325	17,4

The obtained numerical results are consistent with the theoretical convergence estimates established in Theorem 2 and confirm the decrease in error as the approximation parameter tends to zero. The numerical experiment shows that improved accuracy is achieved without a substantial increase in computational effort. Although the computation time increases as ε decreases, this increase remains relatively small compared with the reduction in error. To better understand the behavior of the approximating solution, additional graphical results are presented below. They illustrate the distributions of the main flow variables and provide a visual representation of the approximation error over the computational domain.

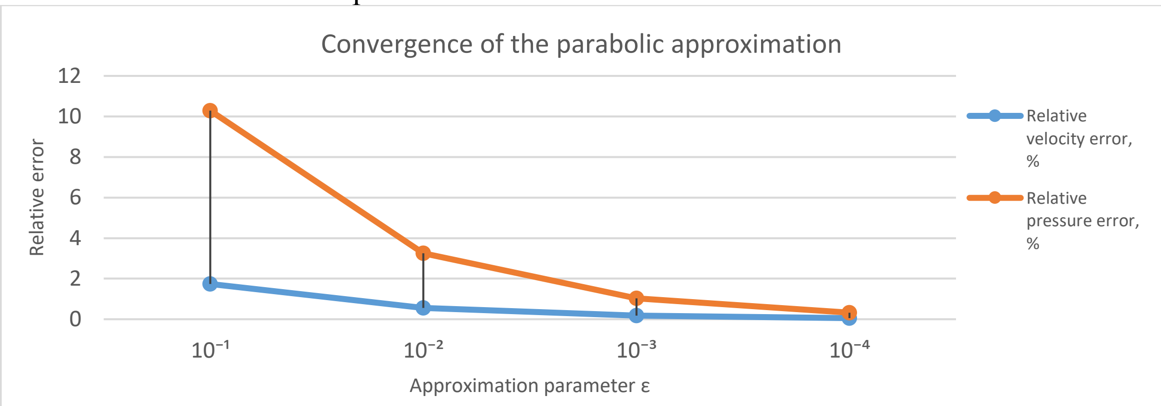

Figure 1. Variation of relative velocity and pressure errors with respect to the approximation parameter ε

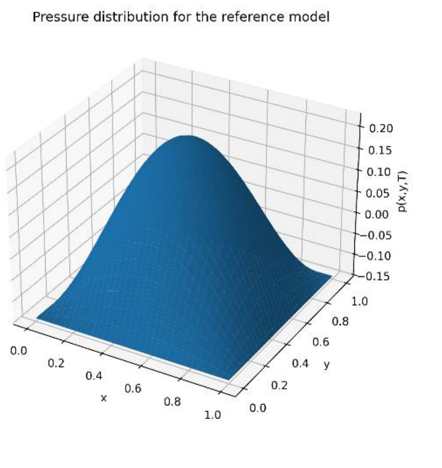

Figure 2. Spatial distribution of pressure in the model problem

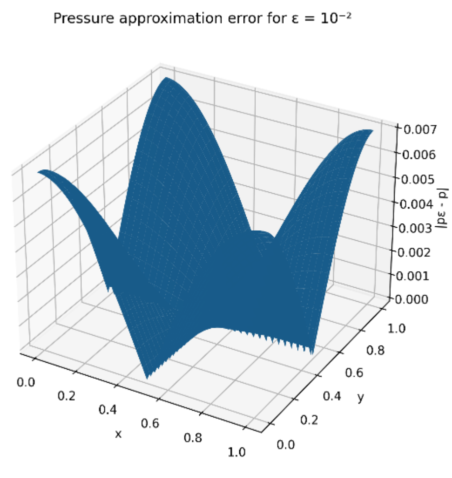

Figure 3. Pressure approximation error surface for $\varepsilon = 10^{-2}$

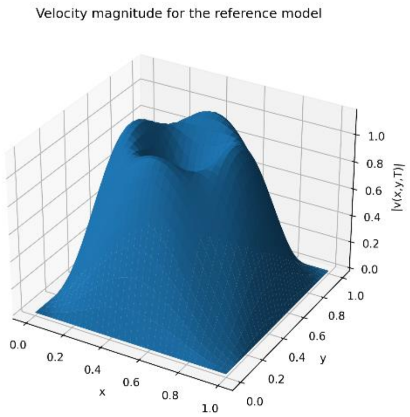

Figure 4. Spatial distribution of the velocity module in the model problem

The numerical results are consistent with the graphical representations shown in Figures 1–4. As the approximation parameter ε decreases, both velocity and pressure errors become smaller. The pressure and velocity fields remain smooth throughout the computational domain, while the approximation error remains relatively small. These observations are in agreement with the theoretical convergence estimates obtained for the approximating system.

Quantitative Estimates of the Model Accuracy

The convergence estimates obtained in the theoretical part of the study make it possible to evaluate the difference between the solution of the original Navier–Stokes problem and the solution of the approximating system. The behavior of this difference is determined by the regularization parameter ε and reflects the rate at which the approximation approaches the original model.

As part of the study, estimates were obtained

$$\int_0^T \|\vec{v}^{\varepsilon}-\vec{v}\|^2_{W_2^1(\Omega)}\,dt+\max_{0\le t\le T}\|\vec{v}^{\varepsilon}-\vec{v}\|^2_{L_2(\Omega)}\le C_3\cdot\varepsilon^{3/2},$$

and

$$\|\vec{v}^{\varepsilon}-\vec{v}\|_{W_2^{2,1}(\mathrm{Q_T})}\le C_4\cdot\varepsilon^{1/2},$$

where the constants C_3 and C_4 are independent of the approximation parameter. The estimates show that the discrepancy between the two solutions decreases as the parameter ε tends to zero. Moreover, they provide information about the convergence rate in the corresponding functional spaces and characterize the influence of the regularization parameter on the approximation accuracy.

The theoretical estimates were compared with the results of a numerical model experiment. The calculations show that reducing ε from 10^{-1} to 10^{-4} leads to a noticeable reduction in both velocity and pressure errors. In particular, the relative velocity error decreases from 1.738% to 0.055%, while the pressure error decreases from 10.285% to 0.325%. At the same time, the increase in computational time remains relatively small.

The quantitative estimates obtained confirm that the parabolic approximation provides a high degree of agreement with the original Navier–Stokes model, even for relatively small values of the ε parameter. This allows the considered approach to be used in computational fluid dynamics problems where a compromise between modeling accuracy and computational efficiency is required. These findings indicate that the proposed approximation can be used in numerical simulations where both accuracy and computational efficiency are important [11, 12].

Results of the Study

This study examined a parabolic approximation of the linear nonstationary Navier–Stokes problem for an incompressible fluid. The analysis allowed us to establish the fundamental properties of the solutions of the approximating system and to substantiate its applicability to the study of nonstationary hydrodynamic processes.

First, a priori estimates were obtained to ensure the existence and uniqueness of a generalized solution to the problem under consideration. It was shown that when the appropriate conditions on the initial data are satisfied, the solution of the approximating system belongs to the required functional spaces and satisfies energy estimates that are independent of the specific implementation of the method. The results confirm the mathematical correctness of the proposed formulation and provide a basis for further analysis.

Further research focused on the regularity of the solution. It was found that with additional smoothness of the initial data, the generalized solution becomes strong, and its derivatives satisfy the corresponding estimates in Sobolev spaces. This demonstrates that the solution's required analytical properties are preserved when transitioning from the original system to its parabolic approximation.

The key result of the study is the derivation of convergence estimates between the solutions of the original problem and the approximating system. To this end, an auxiliary boundary value problem for the difference between the solutions was formulated, and energy inequalities were derived, thereby enabling estimation of the error in terms of the approximation parameter ε. It was shown that as this parameter decreases, the solution to the approximating problem converges to the solution of the original Navier–Stokes system in the corresponding functional norms. The estimates obtained allow us to quantitatively characterize the convergence rate and confirm the correctness of the approach used.

The practical significance of the results is confirmed by the feasibility of using parabolic approximation in numerical modeling of unsteady flows in engineering systems. The theoretical estimates obtained provide a basis for developing robust computational algorithms that strike an acceptable balance between calculation accuracy and computational cost.

The scientific novelty of this study lies in the construction and analysis of a parabolic approximation of the linear nonstationary Navier–Stokes problem, obtained by replacing the incompressibility condition with an evolutionary equation for pressure, with a small regularization parameter. Unlike traditional approaches, which focus primarily on the numerical implementation of artificial compressibility methods and projection schemes, this paper provides a rigorous analytical justification for the approximation under consideration. Conditions for the existence and uniqueness of a generalized solution are established, its regularity properties are investigated, and quantitative estimates of the convergence of the approximating system to the original problem as the parameter ε tends to zero are obtained. An additional result is a computational experiment that confirmed the consistency of theoretical estimates with the observed error behavior. These results broaden the applicability of parabolic approximations to the study of nonstationary hydrodynamic processes and can serve as a theoretical basis for developing new computational algorithms.

Discussion

The proposed approach demonstrates the potential of using parabolic approximation as a tool for studying unsteady-state fluid dynamics problems. Replacing the incompressibility condition with an evolutionary equation for pressure yields a more analytically convenient formulation while preserving the connection to the original physical model. This approach opens up additional possibilities for applying methods from the theory of parabolic equations and obtaining analytical characteristics of solutions.

From a practical standpoint, an important advantage of the approximation under consideration is the reduced complexity of computational procedures. In problems involving repeated modeling of unsteady-state flow regimes, using an approximating system can reduce computational costs without significantly degrading accuracy. This is especially relevant for computational fluid dynamics problems that require a large number of calculations with varying initial or boundary conditions [13, 14]. It should be noted that the results obtained pertain to a linear model, which is considered the first stage in the study of more complex hydrodynamic systems. The linear formulation allows us to focus on the properties of the approximation itself and establish its fundamental characteristics independent of nonlinear effects. However, real fluid flows are often described by a full nonlinear Navier-Stokes system, necessitating further development of this approach.

A promising direction for subsequent research is to extend the developed scheme to nonlinear models of viscous incompressible fluid motion and to develop numerical methods based on the theoretical results obtained. Of additional interest is the study of the behavior of the approximation in problems with complex domain geometry, variable coefficients, and more general boundary conditions. This will expand the method's scope and enhance its practical significance for solving modern problems in mathematical physics and engineering modeling [15].

Conclusion

The completed study substantiated the use of parabolic approximation for the analysis of a model nonstationary Navier-Stokes problem. The approach considered enables a transition from a system subject to a constraint on the divergence of the velocity field to a more convenient formulation, thereby allowing the use of methods from the theory of parabolic equations.

The obtained results demonstrate that the introduction of an approximation parameter does not lead to a loss of the key qualitative characteristics of the solution. The constructed model is mathematically consistent with the original problem and enables rigorous estimates of the solution's behavior in the corresponding functional spaces. The established relationships between the magnitude of the approximation parameter and the solution's accuracy enable quantitative control over the degree of approximation to the original system.

The practical value of this study lies in the potential of the results obtained to develop computational methods for solving non-stationary fluid dynamics problems. The theoretical justification for the convergence and stability of the approximating model provides a foundation for constructing efficient algorithms that reduce computational costs while maintaining the required accuracy.

Thus, the analysis confirms the potential of using parabolic approximations in studying problems in mathematical physics and computational fluid dynamics. The obtained results can serve as a basis for further development of approximation methods and their application to more complex nonlinear models of viscous incompressible fluid motion.

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