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FINITE ELEMENT METHOD FOR SOLVING A FRACTIONAL FLOW MODEL IN POROUS MEDIA

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Abstract

The problem of numerical implementation of multiphase filtration models in highly porous fractured formations is of great applied importance in the oil industry. In this paper, we consider the equation of filtration of a viscoelastic fluid in a fractured porous medium with fractional time derivatives in the sense of Caputo. For the numerical solution, an approximation was constructed using the finite difference method for integer and fractional time derivatives and the finite element method with respect to the spatial variable. The stability a priori estimates for the numerical method with respect to the initial data and the right-hand side of the equation were obtained. The convergence of the constructed method in the spatial direction with the second order of accuracy and in the time variable with the accuracy order of $\min\{2-\alpha,2-\beta,2-\gamma\}$, where $\alpha,\beta,\gamma\in(0,1)$ are the orders of fractional derivatives. The results of numerical tests for the model problem are presented, which show the efficiency of the proposed method for modeling the flow in fractured porous media.

Keywords: finite element method, filtration problem, fractional derivative in the sense of Caputo, fractured porous medium, a priori estimates, stability, convergence.

Аңдатпа

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КЕУЕКТІ ОРТАДАҒЫ АҒЫННЫҢ БӨЛШЕК МОДЕЛІН ШЕШУГЕ АРНАЛҒАН АҚЫРЛЫ ЭЛЕМЕНТТЕР ӘДІСІ

Кеуектілігі жоғары жарықшалы қабаттарда көп фазалы фильтрация модельдерін сандық жүзеге асыру мәселесі мұнай өнеркәсібінде үлкен қолданбалы мәнге ие. Бұл жұмыста Капуто мағынасындағы бөлшек туындылары бар жарықшалы кеуекті ортадағы тұтқыр-серпімді сұйықтықтың фильтрация теңдеуі қарастырылады. Сандық шешім үшін бүтін және бөлшек уақыт туындылары үшін ақырлы айырымдар әдісін, ал кеңістіктік айнымалысы бойынша ақырлы элементтер әдісін қолдана отырып жуықтау жасалды. Бастапқы мәліметтер және теңдеудің оң жағы бойынша сандық әдістің орнықтылығы үшін априорлық бағалау алынды. Құрылған әдістің кеңістіктік бағыт бойынша екінші ретті дәлдікпен және уақыт айнымалысы бойынша $\min\{2-\alpha,2-\beta,2-\gamma\}$ ретті дәлдікпен жинақталуы дәлелденді, мұндағы $\alpha,\beta,\gamma\in(0,1)$ – бөлшек туындылардың реті. Жарықшалы кеуекті ортадағы сұйықтық ағынының моделі үшін ұсынылған әдісінің тиімділігін көрсететін үлгілік есепті сандық шешудің нәтижелері келтірілген.

Түйін сөздер: ақырлы элементтік әдіс, фильтрация есебі, Капуто мағынасындағы бөлшек туынды, жарықшалы кеуекті орта, априорлық бағалау, тұрақтылық, жинақтылық.

Аннотация

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МЕТОД КОНЕЧНЫХ ЭЛЕМЕНТОВ ДЛЯ РЕШЕНИЯ ДРОБНОЙ МОДЕЛИ ТЕЧЕНИЯ В ПОРИСТЫХ СРЕДАХ

Проблема численной реализации моделей многофазной фильтрации в сильнопористых трещиноватых пластах имеет большое прикладное значение в нефтедобывающей промышленности. В настоящей работе рассматривается уравнение фильтрации вязкоупругой жидкости в трещиновато-пористой среде с дробными производными по времени в смысле Капуто. Для численного решения построена аппроксимация с использованием метода конечных разностей для целых и дробных временных производных и метода конечных элементов по пространственной переменной. Получены априорные оценки устойчивости численного метода по начальным данным и по правой части уравнения. Доказана сходимость построенного метода по пространственному направлению со вторым порядком точности и по временной переменной с порядком точности $\min\{2-\alpha, 2-\beta, 2-\gamma\}$, $\alpha, \beta, \gamma \in (0,1)$ – порядки дробных производных. Представлены результаты численного решения модельной задачи, которые показывают эффективность предложенного метода для моделирования течения в трещиноватых пористых средах.

Ключевые слова: конечно-элементный метод, задача фильтрации, дробная производная в смысле Капуто, трещиноватая пористая среда, априорные оценки, устойчивость, сходимость.

1 Introduction

The study of filtration processes in fractured formations is an urgent area of modern computational fluid dynamics both from the aspect of constructing new mathematical models of the fluid flow process applicable for real problems, and from the aspect of constructing modern numerical algorithms [1, 2]. As recent studies show, the processes of filtration flows, where their dynamics are influenced by a fractured-porous medium and memory effects, are more realistically described using the theory of fractional-order integro-differentiation [3-6]. To describe such processes, the fractional derivative in the sense of Caputo was used to take into account the memory formalism [7]. The fractional derivative in the Caputo-Fabrizio sense was used to account for longitudinal dispersion in a flow of two miscible fluids [8]. In [3, 9], the Darcy's law was reformulated with the help of a fractional derivative in the sense of Riemann-Liouville.

Much research is being done on numerical methods for solving fluid flow problems. Due to the need for calculations in complex geometric areas, the finite element method was used in [10, 11]. In [12], an extended mixed finite element method was used for the spatial fractional Darcy flow in porous media. In our previous work [13], finite difference schemes were constructed for solving initial-boundary value problems for the convection-diffusion equation with a fractional time derivatives in the sense of Riemann-Liouville, which can be used as a model filtration problem.

The purpose of this work is to study the finite difference/finite element method for implementing the model of filtration in a fractured porous medium proposed in [3], as well as to study the stability with respect to the initial data and the right-hand side of the equation. The results of computational experiments carried out for various orders of fractional derivatives and grid configurations are presented.

2 Formulation of the Problem

In [3], a model of a single-phase flow of an isothermal fluid in a homogeneous porous medium is presented by replacing the porosity function, the equation of state and Darcy's law with their fractional-differential analogs:

$$\phi\left(c_{f1}+c_{\phi 1}\right)\frac{\partial u}{\partial t}+\phi c_{\phi\alpha}\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+\phi c_{f\beta}\frac{\partial^{\beta}u}{\partial t^{\beta}}-$$

$$-\frac{F\left(\dfrac{\partial^{\gamma}(\nabla u)}{\partial t^{\gamma}}\right)}{\phi}\left(c_{f1}\nabla u+c_{f\beta}\frac{\partial^{\beta}(\nabla u)}{\partial t^{\beta}}\right)\frac{\nabla u}{|\nabla u|}-\nabla\cdot\left(\frac{F\left(\dfrac{\partial^{\gamma}(\nabla u)}{\partial t^{\gamma}}\right)}{|\nabla u|}\nabla u\right)=f_0, \tag{1}$$

where $f_0 = \dfrac{f}{\rho}$; $c_{f1} = \dfrac{1}{\rho}\dfrac{\partial \rho}{\partial u}$, $c_{\phi 1} = \dfrac{1}{\phi}\dfrac{\partial \phi}{\partial u}$ are classical isothermal compressibility of fluid and porous medium, and $c_{f\beta}$, $c_{\phi\alpha}$ are the generalized fractional differential isothermal compressibility [3].

In this paper, we mainly focus on studying a special one-dimensional case of the model (1) under the assumption that the pressure gradient is small, which is often valid for filtration processes. In domain $\overline{Q}_T = \overline{\Omega} \times [0,T]$, where $\Omega = (0,1)$ find u satisfying

$$\begin{cases} \dfrac{\partial u}{\partial t} + \bar{c}_{\phi\alpha}\dfrac{\partial^\alpha u}{\partial t^\alpha} + \bar{c}_{f\beta}\dfrac{\partial^\beta u}{\partial t^\beta} - \left(F\left(\dfrac{\partial^\gamma u_x}{\partial t^\gamma}\right)\right)_x = \bar{f}_0, & x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \\ u(0,t) = u(1,t) = 0, & t > 0, \end{cases} \tag{2}$$

where $\alpha, \beta, \gamma \in (0,1)$.

Throughout this paper, the following assumptions will be used:

(I) Problem 2 has a unique solution that has the number of derivatives required for the analysis.

(II) F is a linear operator defined on Ω such that

$$F(\varphi) = \mu\varphi(x,t), \tag{3}$$

where $\mu > 0$ is a constant.

Let us define a variational formulation of Problem 2. Find $u \in H^1(0,T;H_0^1(\Omega))$ such that for any $v \in H_0^1(\Omega)$:

$$\left(\frac{\partial u}{\partial t}, v\right) + \bar{c}_{\phi\alpha}\left(\frac{\partial^\alpha u}{\partial t^\alpha}, v\right) + \bar{c}_{f\beta}\left(\frac{\partial^\beta u}{\partial t^\beta}, v\right) + \left(F\left(\frac{\partial^\gamma u_x}{\partial t^\gamma}\right), v_x\right) = \left(\bar{f}_0, v\right), \tag{4}$$

$$u(x,0) = u_0(x), \ x \in \overline{\Omega}, \tag{5}$$

where $\alpha, \beta, \gamma \in (0,1)$.

On the time interval $[0,T]$, we introduce the finite difference grid

$$\omega_\tau = \{t_n = n\tau, n = 0,1,\ldots, N, N\tau = T\}.$$

We denote by u^n the semi-discrete approximation of the function u with respect to time at point $t = t_n$. We use the formula [14]

$$\Delta^\nu u(t_n) = \sum_{s=1}^{n} \delta_{n,s}^\nu \left(u^s - u^{s-1}\right), \tag{6}$$

approximating the fractional derivative of order $0 < \nu < 1$ in the sense of Caputo with the order of approximation $O(\tau^{2-\nu})$, where

$$\delta_{n,s}^\nu = \frac{\tau^{-\nu}}{\Gamma(2-\nu)}\left[(n-s+1)^{1-\nu} - (n-s)^{1-\nu}\right].$$

Let the values $u^i \in H_0^1(\Omega)$, $i = 0,1,\ldots,n-1$ be known. Find $u^n \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$:

a) when $n=1$:

$$\left(\frac{u^1-u^0}{\tau},v\right)+\bar{c}_{\phi\alpha}\left(\Delta^\alpha u^1,v\right)+\bar{c}_{f\beta}\left(\Delta^\beta u^1,v\right)+\left(F\left(\Delta^\gamma u_x^1\right),v_x\right)=\left(\bar{f}_0,v\right); \quad (7)$$

b) when $n\geq 2$:

$$\left(\frac{3u^n-4u^{n-1}+u^{n-2}}{2\tau},v\right)+\bar{c}_{\phi\alpha}\left(\Delta^\alpha u^n,v\right)+\bar{c}_{f\beta}\left(\Delta^\beta u^n,v\right)+$$
$$+\left(F\left(\Delta^\gamma u_x^n\right),v_x\right)=\left(\bar{f}_0,v\right), \quad (8)$$

$$u^0=u_0(x),\ x\in\overline{\Omega}, \quad (9)$$

where $\alpha,\beta,\gamma\in(0,1)$.

Now let us define a fully discrete formulation of Problem (4)-(5). In $\overline{\Omega}$, introduce a quasi-uniform triangulation K_h. Let $Q_h{:}H_0^1(\Omega)\to V_h$ denote the elliptic projection. Let the values $u_h^i\in V_h$, $i=0,1,...,n-1$ be known, where

$$V_h=\left\{v_h\in H_0^1(\Omega)\cap C^0\left(\overline{\Omega}\right)\middle| v_h|_e\in P_1(e),\forall e\in K_h\right\},$$

$P_l(e)$ is the space of polynomials of degree at most l on $e\in K_h$. Find $u_h^n\in V_h$ satisfying the following identities for any $v_h\in V_h$:

a) when $n=1$:

$$\left(\frac{u_h^1-u_h^0}{\tau},v_h\right)+\bar{c}_{\phi\alpha}\left(\Delta^\alpha u_h^1,v_h\right)+\bar{c}_{f\beta}\left(\Delta^\beta u_h^1,v_h\right)+$$
$$+\left(F\left(\Delta^\gamma u_{h,x}^1\right),v_{h,x}\right)=\left(\bar{f}_0,v_h\right), \quad (10)$$

b) when $n\geq 2$:

$$\left(\frac{3u_h^n-4u_h^{n-1}+u_h^{n-2}}{2\tau},v_h\right)+\bar{c}_{\phi\alpha}\left(\Delta^\alpha u_h^n,v_h\right)+\bar{c}_{f\beta}\left(\Delta^\beta u_h^n,v_h\right)+$$
$$+\left(F\left(\Delta^\gamma u_{h,x}^n\right),v_{h,x}\right)=\left(\bar{f}_0,v_h\right), \quad (11)$$

$$u_h^0=Q_h u_0, \quad (12)$$

where $\alpha,\beta,\gamma\in(0,1)$.

3 A Priori Estimates

To obtain a priori estimates, we use the following auxiliary lemma. For any function $u^n\in L^2(\Omega)$, the following inequality holds [2]:

$$\left(\Delta^\nu u^n,u^n\right)\geq\Theta_n^\nu-\Theta_{n-1}^\nu-\frac{1}{2}\delta_{n,1}^\nu\left\|u^0\right\|_0^2,$$

where $\Theta_i^\nu=\frac{1}{2}\left(\delta_{i,1}^\nu\left\|u^1\right\|_0^2+\delta_{i,2}^\nu\left\|u^2\right\|_0^2+...+\delta_{i,i}^\nu\left\|u^i\right\|_0^2\right)$, $i\geq 1$ and $\Theta_0^\nu=0$.

The discrete scheme (10)-(12) is unconditionally stable with respect to the initial data and right-hand side, and the following estimate holds:

$$\left\|u_h^n\right\|_0^2 + 8\tau\Theta_n \le C\left(\left(\tau^2 + T\right)\left\|\bar{f}_0\right\|_0^2 + \left\|u_h^0\right\|_1^2\right).$$

Proof.

a) Let us choose $v_h = u_h^1$ in (10). Then using (6), we obtain

$$\left(\frac{u_h^1 - u_h^0}{\tau}, u_h^1\right) + \bar{c}_{\phi\alpha}\left(\delta_{1,1}^{\alpha}\left(u_h^1 - u_h^0\right), u_h^1\right) + \bar{c}_{f\beta}\left(\delta_{1,1}^{\beta}\left(u_h^1 - u_h^0\right), u_h^1\right) +$$
$$+\left(F\left(\delta_{1,1}^{\gamma}\left(u_h^1 - u_h^0\right)\right), u_{h,x}^1\right) = \left(\bar{f}_0, u_h^1\right).$$

Making use of Lemma 3 and considering (3), we get

$$\left\|u_h^1\right\|_0^2 + \left\|u_h^1 - u_h^0\right\|_0^2 + 2\tau\left(\bar{c}_{\phi\alpha}\Theta_1^{\alpha} + \bar{c}_{f\beta}\Theta_1^{\beta} + \mu\Theta_1^{\gamma}\right) \le$$
$$\le 2\tau\left(\bar{f}_0, u_h^1\right) + \left\|u_h^0\right\|_0^2 + 2\tau\left(\bar{c}_{\phi\alpha}\Theta_0^{\alpha} + \bar{c}_{f\beta}\Theta_0^{\beta} + \mu\Theta_0^{\gamma}\right) +$$
$$+\tau\left(\bar{c}_{\phi\alpha}\delta_{1,1}^{\alpha} + \bar{c}_{f\beta}\delta_{1,1}^{\beta}\right)\left\|u_h^0\right\|_0^2 + \tau\mu\delta_{1,1}^{\gamma}\left\|u_{h,x}^0\right\|_0^2.$$

By applying the Cauchy and Young inequalities to the first term on the right-hand side, we obtain:

$$\left\|u_h^1\right\|_0^2 + 2\left\|u_h^1 - u_h^0\right\|_0^2 + 4\tau\Theta_1 \le 4\tau^2\left\|\bar{f}_0\right\|_0^2 + C\left\|u_h^0\right\|_1^2, \tag{13}$$

where $\Theta_n = \bar{c}_{\phi\alpha}\Theta_n^{\alpha} + \bar{c}_{f\beta}\Theta_n^{\beta} + \mu\Theta_n^{\gamma}$, $C = \max\left\{2 + T\left(\bar{c}_{\phi\alpha}\delta_{1,1}^{\alpha} + \bar{c}_{f\beta}\delta_{1,1}^{\beta}\right), T\mu\delta_{1,1}^{\gamma}\right\}$.

b) In case of $n \ge 2$, choose $v_h = u_h^n$ in (11) and utilize (6) to obtain

$$\left(\frac{3u_h^n - 4u_h^{n-1} + u_h^{n-2}}{2\tau}, v_h\right) + \bar{c}_{\phi\alpha}\left(\sum_{s=1}^{n}\delta_{n,s}^{\alpha}\left(u_h^s - u_h^{s-1}\right), v_h\right) +$$
$$+\bar{c}_{f\beta}\left(\sum_{s=1}^{n}\delta_{n,s}^{\beta}\left(u_h^s - u_h^{s-1}\right), v_h\right) + \left(F\left(\sum_{s=1}^{n}\delta_{n,s}^{\gamma}\left(u_{h,x}^s - u_{h,x}^{s-1}\right)\right), v_{h,x}\right) = \left(\bar{f}_0, v_h\right).$$

By applying Lemma 3 and using the assumption (3), we arrive at the inequality

$$\left\|u_h^n\right\|_0^2 + \left\|2u_h^n - u_h^{n-1}\right\|_0^2 + \left\|u_h^n - 2u_h^{n-1} + u_h^{n-2}\right\|_0^2 +$$
$$+4\tau\Theta_n = 4\tau\left\|\bar{f}_0\right\|_0\left\|u_h^n\right\|_0 + \left\|u_h^{n-1}\right\|_0^2 + \left\|2u_h^{n-1} - u_h^{n-2}\right\|_0^2 +$$
$$+4\tau\Theta_{n-1} + 2\tau\left(\bar{c}_{\phi\alpha}\delta_{n,1}^{\alpha} + \bar{c}_{f\beta}\delta_{n,1}^{\beta}\right)\left\|u_h^0\right\|_0^2 + 2\tau\mu\delta_{n,1}^{\gamma}\left\|u_{h,x}^0\right\|_0^2.$$

Sum the last inequality with respect to n from 2 to n to get

$$\left\|u_h^n\right\|_0^2 + 4\tau\Theta_n = 4\tau\left\|\bar{f}_0\right\|_0\left\|u_h^n\right\|_0 + 4\tau\sum_{i=2}^{n-1}\left\|\bar{f}_0\right\|_0\left\|u_h^i\right\|_0 + \left\|u_h^1\right\|_0^2 + \left\|2u_h^1 - u_h^0\right\|_0^2 +$$
$$+4\tau\Theta_1 + 2\tau\sum_{i=2}^{n}\left(\bar{c}_{\phi\alpha}\delta_{i,1}^{\alpha} + \bar{c}_{f\beta}\delta_{i,1}^{\beta}\right)\left\|u_h^0\right\|_0^2 + 2\tau\mu\sum_{i=2}^{n}\delta_{i,1}^{\gamma}\left\|u_{h,x}^0\right\|_0^2,$$

and taking into account the elementary inequality $ab \le \frac{1}{2}\left(a^2 + b^2\right)$ we arrive at

$$\left\|u_h^n\right\|_0^2+8\tau\Theta_n\le\left(16\tau^2+4T\right)\left\|\bar f_0\right\|_0^2+4\tau\sum_{i=2}^{n-1}\left\|u_h^i\right\|_0^2+$$
$$+\left\|u_h^1\right\|_0^2+\left\|2u_h^1-u_h^0\right\|_0^2+4\tau\Theta_1+$$
$$+2\tau\sum_{i=2}^{n}\left(\bar c_{\phi\alpha}\delta_{i,1}^{\alpha}+\bar c_{f\beta}\delta_{i,1}^{\beta}\right)\left\|u_h^0\right\|_0^2+2\tau\mu\sum_{i=2}^{n}\delta_{i,1}^{\gamma}\left\|u_{h,x}^0\right\|_0^2.$$

Making use of the discrete Gronwall's lemma, we obtain

$$\left\|u_h^n\right\|_0^2+8\tau\Theta_n\le C\Bigg(\left(16\tau^2+4T\right)\left\|\bar f_0\right\|_0^2+3\left(\left\|u_h^1\right\|_0^2+2\left\|u_h^1-u_h^0\right\|_0^2+4\tau\Theta_1\right)+$$
$$+2\tau\sum_{i=2}^{n}\left(\bar c_{\phi\alpha}\delta_{i,1}^{\alpha}+\bar c_{f\beta}\delta_{i,1}^{\beta}\right)\left\|u_h^0\right\|_0^2+2\tau\mu\sum_{i=2}^{n}\delta_{i,1}^{\gamma}\left\|u_{h,x}^0\right\|_0^2\Bigg). \qquad (14)$$

Combining (13) and (14), we arrive at the assertion of the theorem.

Let us present the convergence results the proof of which are presented in [2]. Let $\{u^i\}_{i=0}^N$, $u^i\in H_0^1(\Omega)$ be the solution of Problem (7)-(9). Then, under assumptions (I)-(II) and $\alpha,\beta,\gamma\in(0,1)$, the following inequality holds:

$$\left\|u(t_n)-u^n\right\|_0+\tau\sqrt{\frac{2c_0}{T}}\left\|u(t_n)-u^n\right\|_1\le C\tau^{\min\{2-\alpha,2-\beta,2-\gamma\}},$$

where $c_0=\min\{\bar c_{\phi\alpha}\delta_{n,1}^{\alpha},\bar c_{f\beta}\delta_{n,1}^{\beta},\mu\delta_{n,1}^{\gamma}\}$.

Let $\{u_h^i\}_{i=0}^N$, $u_h^i\in V_h$ be the solution of Problem (10)-(12). Then, under assumptions (I)-(II) and $\alpha,\beta,\gamma\in(0,1)$ for $u_h^n\in V_h$ and sufficiently small τ, the following inequality holds:

$$\left\|u(t_n)-u_h^n\right\|_0+\tau\sqrt{\frac{2c_0}{T}}\left\|u(t_n)-u_h^n\right\|_1\le C\left(\tau^{\min\{2-\alpha,2-\beta,2-\gamma\}}+h^2\right),$$

where $c_0=\min\{\bar c_{\phi\alpha}\delta_{n,1}^{\alpha},\bar c_{f\beta}\delta_{n,1}^{\beta},\mu\delta_{n,1}^{\gamma}\}$.

4 Implementation of the Method and Computational Experiments

In this section, we present the results of numerical experiments to verify the theoretical estimates obtained. We seek the solution in the form

$$u_h^n=\sum_{j=1}^{NN}u_j^n v_j(x),$$

where $v_j(x)$ are basis functions, and choose $v_h=v_i(x)$. Then taking into account (3), we get

$$\left(\Delta_t u_j^n,v_i\right)+\sum_{j=1}^{NN}\left[\left(\bar c_{\phi\alpha}\delta_{n,n}^{\alpha}+\bar c_{f\beta}\delta_{n,n}^{\beta}\right)\left(v_j,v_i\right)+\mu\delta_{n,n}^{\gamma}\left(v_{j,x},v_{i,x}\right)\right]u_j^n=$$
$$=\left(\bar f_0,v_i\right)+\sum_{j=1}^{NN}\left[\left(\bar c_{\phi\alpha}\delta_{n,n}^{\alpha}+\bar c_{f\beta}\delta_{n,n}^{\beta}\right)\left(v_j,v_i\right)+\mu\delta_{n,n}^{\gamma}\left(v_{j,x},v_{i,x}\right)\right]u_j^{n-1}-$$
$$-\sum_{j=1}^{NN}\sum_{s=1}^{n-1}\left[\bar c_{\phi\alpha}\delta_{n,s}^{\alpha}+\bar c_{f\beta}\delta_{n,s}^{\beta}\right]\left(u_j^s-u_j^{s-1}\right)\left(v_j,v_i\right)-$$
$$-\sum_{j=1}^{NN}\sum_{s=1}^{n-1}\mu\delta_{n,s}^{\gamma}\left(u_j^s-u_j^{s-1}\right)\left(v_{j,x},v_{i,x}\right).$$

where $\Delta_t u_j^n$ is the approximation of the integer derivative with respect to time used in Problem 2.

To test the theoretical convergence estimates obtained in Theorem 3, a number of computational experiments were carried out for a model problem. Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t}+\frac{\partial^{\alpha} u}{\partial t^{\alpha}}+\frac{\partial^{\beta} u}{\partial t^{\beta}}-\frac{\partial^{\gamma} u_{xx}}{\partial t^{\gamma}}=8\sin \pi x(2t-1)(4t^2-4t+1)+$$

$$+8\sin \pi x\left(t^{1-\alpha} f(\alpha,t)+t^{1-\beta} f(\beta,t)+\pi^2 t^{1-\gamma} f(\gamma,t)\right),\ x\in\Omega, t>0,$$

$$u(x,0)=0,\ \ x\in\overline{\Omega},$$

$$u(0,t)=u(1,t)=0,\ \ t>0,$$

where $\alpha,\beta,\gamma\in(0,1)$, $f(\nu,t)=\dfrac{\nu^3+6t\nu^2-9\nu^2+24t^2\nu-42t\nu+26\nu+48t^3-96t^2+72t-24}{\Gamma(1-\nu)(\nu-4)(\nu-3)(\nu-2)(\nu-1)}$.

The exact solution to this problem is $u(x,t)=(4t^2-4t+1)^2\sin(\pi x)$.

In the computational experiments, we consider different values of τ ranging from 1/10 to 1/160 with a fixed grid size $h=1/20000$. It follows from Table 0 that the convergence order decreases with increasing values of the fractional derivative's orders. Specifically, the scheme achieved the convergence order of 1.90 when $\alpha=\beta=\gamma=0.1$ and the convergence order of 1.10 when any of α,β,γ is equal to 0.9. This result is consistent with the theoretical estimates indicated in parentheses in the «Order» column.

Table 1. L^2 *-errors and convergence order for Example*

γ	τ	$\alpha=\beta=0.1$		$\alpha=\beta=0.5$		$\alpha=\beta=0.9$	
		L^2-error	Order	L^2-error	Order	L^2-error	Order
$\gamma=0.1$	1/10	1.3980e-02	-	2.0710e-02	-	5.2335e-02	-
	1/20	4.0621e-03	1.78 (≈ 1.90)	6.8566e-03	1.59 (≈ 1.50)	2.3283e-02	1.17 (≈ 1.10)
	1/40	1.1228e-03	1.86 (≈ 1.90)	2.2250e-03	1.62 (≈ 1.50)	1.0490e-02	1.15 (≈ 1.10)
	1/80	3.0352e-04	1.89 (≈ 1.90)	7.2301e-04	1.62 (≈ 1.50)	4.7846e-03	1.13 (≈ 1.10)
	1/160	8.1199e-05	1.90 (≈ 1.90)	2.3719e-04	1.61 (≈ 1.50)	2.2017e-03	1.12 (≈ 1.10)
$\gamma=0.5$	1/10	4.2604e-02	-	4.7297e-02	-	7.2854e-02	-
	1/20	1.5517e-02	1.46 (≈ 1.50)	1.7339e-02	1.45 (≈ 1.50)	3.0496e-02	1.26 (≈ 1.10)
	1/40	5.5576e-03	1.48 (≈ 1.50)	6.2503e-03	1.47 (≈ 1.50)	1.2866e-02	1.25 (≈ 1.10)
	1/80	1.9751e-03	1.49 (≈ 1.50)	2.2334e-03	1.48 (≈ 1.50)	5.4951e-03	1.23 (≈ 1.10)
	1/160	6.9940e-04	1.50 (≈ 1.50)	7.9429e-04	1.49 (≈ 1.50)	2.3785e-03	1.21 (≈ 1.10)
$\gamma=0.9$	1/10	1.5551e-01	-	1.5610e-01	-	1.7391e-01	-
	1/20	7.2442e-02	1.10 (≈ 1.10)	7.2310e-02	1.11 (≈ 1.10)	8.1161e-02	1.10 (≈ 1.10)
	1/40	3.3765e-02	1.10 (≈ 1.10)	3.3537e-02	1.11 (≈ 1.10)	3.7882e-02	1.10 (≈ 1.10)
	1/80	1.5750e-02	1.10 (≈ 1.10)	1.5576e-02	1.11 (≈ 1.10)	1.7686e-02	1.10 (≈ 1.10)
	1/160	7.3487e-03	1.10 (≈ 1.10)	7.2419e-03	1.10 (≈ 1.10)	8.2569e-03	1.10 (≈ 1.10)

5 Conclusion

Thus, a finite difference/finite element scheme has been constructed for the fractional differential equation containing several fractional time derivatives in the sense of Caputo. Numerical experiments fully confirm the theoretical estimates obtained in Theorem 3. The results obtained can find application in the numerical solution of other equations containing a fractional time derivative and will form the basis for further research in this direction.

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References:

1 Akkutlu I. Y., Efendiev Y., Vasilyeva M., Wang Y. (2018) Multiscale Model Reduction for Shale Gas Transport in Poroelastic Fractured Media. // J. Comput. Phys., Vol. 353, 356–376. https://doi.org/10.1016/j.jcp.2017.10.023

2 Baigereyev D., Alimbekova N., Berdyshev A., Madiyarov M. (2021) Convergence Analysis of a Numerical Method for a Fractional Model of Fluid Flow in Fractured Porous Media. // Mathematics, Vol. 9 (2179), 1–24. https://doi.org/10.3390/math9182179

3 Gazizov R. K., Lukaschuk S. Yu. (2017) Fractional Differential Approach to Modeling Filtration Processes in Complex Inhomogeneous Porous Media. // Vestnik UGATU Vol. 21(4), 104–112.

4 Hashan M., Jahana T. U., Zaman L. N., Imtiaz S., Hossain M. E. (2020) Modelling of fluid flow through porous media using memory approach: a Review. // Mathematics and Computers in Simulation, Vol. 177, 643–673. https://doi.org/10.1016/j.matcom.2020.05.026

5 Di Giuseppe E., Moroni M., Caputo M. (2010) Flux in Porous Media with Memory: Models And Experiments. // Transport in Porous Media, Vol. 83(3), 479–500. https://doi.org/10.1007/s11242-009-9456-4

6 Abiola O. D., Enamul H. M., Kaseem M., Sidqi A. A. (2017) A Modified Memory-Based Mathematical Model Describing Fluid Flow in Porous Media. // Computers and Mathematics with Applications, Vol. 73 (6), 1385–1402. https://doi.org/10.1016/j.camwa.2016.11.022

7 Caputo M. (2000) Models of Flux in Porous Media with Memory // Water Resources Research, Vol. 36 (3), 693–705. https://doi.org/10.1029/1999WR900299

8 Agarwal R., Yadav M. P., Baleanu D., Purohit S. D. (2020) Existence and Uniqueness of Miscible Flow Equation Through Porous Media with a Non Singular Fractional Derivative. // AIMS Mathematics, Vol. 5 (2), 1062–1073. https://doi.org/10.3934/math.2020074

9 He J. H. (1998) Approximate Analytical Solution for Seepage Flow with Fractional Derivatives in Porous Media. // Computer Methods in Applied Mechanics and Engineering, Vol. 167 (1), 57–88.

10 Ali H., Soleimani H., Yahya N., Baig M. K., Rostami A. (2018) Finite Element Method for Modelling of Two Phase Fluid Flow in Porous Media. // Journal of Physics: Conference Series. http://doi.org/10.1088/1742-6596/1123/1/012002

11 Beybalaev V. D., Abduragimov E. I., Yakubov A. Z., Meilanov R. R., Aliverdiev A. A. (2021) Numerical Research of Non-Isothermal Filtration Process in Fractal Medium with Non-Locality in Time. // Thermal Science, Vol. 25 (1B), 465–475.

12 Chen H., Sun S. (2020) An Expanded Mixed Finite Element Method for Space Fractional Darcy Flow in Porous Media. // Computational Science – ICCS 2020, Vol. 12143, 205–216. https://doi.org/10.1007/978-3-030-50436-6_15

13 Alimbekova N., Berdyshev A., Baigereyev D. (2021) Parallel Implementation of the Algorithm for Solving a Partial Differential Equation with a Fractional Derivative in the Sense of Riemann-Liouville. // 2021 IEEE International Conference on Smart Information Systems and Technologies (SIST), Vol. 9465922, 1–6. http://doi.org/10.1109/SIST50301.2021.9465922

14 Zhang Y. N., Sun Z. Z., Liao H. L. (2014) Finite Difference Methods for the Time Fractional Diffusion Equation on Non-Uniform Meshes. // Journal of Computational Physics, Vol. 265, 195–210. http://doi.org/10.1016/j.jcp.2014.02.008