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# R. KALMAN `S PROBLEM ABOUT FIBONACCI `S NUMBERS

*Iskakova M.T.[1], Shuakayev M.K.[1], Tuiykov E.A.[1], Nazarbekova K.T.[2]*

*[1]Abai Kazakh National Pedagogical University, Almaty, Kazakhstan*
*[2]al-Farabi Kazakh National University, Almaty, Kazakhstan*

*Abstract*

In this paper authors are considered the R. Kalman`s problem about of Fibonacci numbers. An overview of research methods for control theory systems in two concepts “state space” and the “input-output” mapping is presented. In this paper, we consider the problem of R. Kalman on Fibonacci numbers, which consists in the following. R. Kalman's problem on Fibonacci numbers is considered, which is as follows. Fibonacci numbers form a minimal Realization. The authors of the article formulated a theorem, which was given the name of the outstanding American Scientist R. Kalman. The proof of the theorem is very cumbersome, therefore, authors proved it using an example when the Fibonacci numbers are obtained on the basis of the application of the B. Ho`s algorithm. B. Ho is a purple of R. Kalman.

In this paper, the algorithm of B. Ho is given, which allows one to find the parameters of the initial linear deterministic system. Based on these parameters, we find the initial Fibonacci numbers. Thus, Fibonacci numbers are closely related to the problem of linear deterministic implementation and to B. Ho's algorithm.

**Keywords:** Fibonacci numbers, minimal realization, B. Ho algorithm, systems theory, the concept of "states space" and the Map "input - output".

*Аннотация*

*М Т. Искакова[1] М.К. Шуакаев[1], Е.А. Туяков[1], К.Т. Назарбекова[2]*
*[1]Казахский национальный педагогический университет имени Абая, г.Алматы, Казахстан*
*[2]Казахский национальный университет имени аль-Фараби, г.Алматы, Казахстан*

**ПРОБЛЕМА Р. КАЛМАНА О ЧИСЛАХ ФИБОНАЧЧИ**

В работе рассматривается проблема Р. Калмана о числах Фибоначи. Представлен обзор методов исследования систем теории управления в двух концепциях «пространство состояний» и отображение «вход – выход». Рассматривается проблема Р. Калмана о числах Фибоначчи, которая заключается в следующем. Числа Фибоначчи образуют минимальную реализацию. Авторы статьи сформулировали теорему, которой дали имя выдающегося американского ученого Р. Калмана. Доказательство теоремы весьма громоздкое, поэтому авторы доказали ее на примере, когда числа Фибоначчи получаются на основе применения алгоритма Б. Хо – ученика Р. Калмана.

В работе приводится алгоритм Б. Хо, который позволяет найти параметры исходной линейной детерминированной системы. На основании этих параметров мы находим исходные числа Фибоначчи. Тем самым, числа Фибоначчи имеют тесную связь с проблемой линейной детерминированной реализации и с алгоритмом Б. Хо.

**Ключевые слова:** числа Фибоначчи, минимальная реализация, алгоритм Б. Хо, теория систем, концепции «пространство состояний» и отображение «вход – выход».

*Аңдатпа*

*М.Т. Искакова[1], М. Қ. Шуақаев[1], Е.А. Тұяқов[1], К.Т. Назарбекова[2]*
*[1]Абай атындағы Қазақ ұлттық педагогиялық университеті, Алматы қ., Қазақстан*
*[2]әл-Фараби атындағы Қазақ ұлттық университеті, Алматы қ., Қазақстан*

**ФИБОНАЧИ САНДАРЫ ТУРАЛЫ Р. КАЛМАННЫҢ МӘСЕЛЕСІ**

Мақалада Фибоначчи сандары жайлы Р.Кальман мәселесі қарастырылған. Бұл мақалада «кеңістік жағдайы» және «енгізу-шығару» бейнеленуі екі тұжырымдамасындағы басқару теориясының жүйелерін зерттеу әдістеріне шолу келтірілген. Фибоначчи сандары туралы Р.Кальман мәселесі қарастырылады, ол төменде келтіріледі. Фибоначчи сандары минималды енгізуді құрайды. Мақала авторлары көрнекті американдық ғалым Р.Кальманның есімімен аталатын теореманы тұжырымдады. Теореманың дәлелі өте қиын, сондықтан авторлар Фибоначчи сандарын Б.Хо-шәкірті Р.Кальманның алгоритмінің негізінде алынған кезде мысал келтіріп дәлелдеді. Бұл жұмыста бастапқы сызықтық детерминистикалық жүйенің параметрлерін табуға мүмкіндік беретін Б. Хо алгоритмі келтірілген. Осы параметрлер негізінде Фибоначчидің бастапқы сандарын табамыз.

Сонымен, Фибоначчи сандары сызықтық детерминистік іске асыру мәселесімен және Б.Хо алгоритмімен тығыз байланысты.

**Түйін сөздер:** Фибоначчи сандары, минималды енгізу, Б. Хо алгоритмі, жүйелер теориясы, «кеңістік жағдайы» тұжырымдамасы және «кіру-шығу» бейнелеуі.

In the theory of Volterra's series exists three main directions of their research. The first [1-4] – uses a method of the analysis of the "Space of States" transformation which is based on the description of system on the base of some functionality. It allows considering system as converter of the process operating on it's into output process. There are two concepts of the description r investigation of mathematical models in conception of the transformation "Space of States". In this case, it isn't required information about internal structure of system; its properties are investigated in terms of global characteristics of systems as whole. The second [1-4] the deterministic (stochastic) system the differential ordinary (stochastic) equations on the basis of the known Map "Input - Output" presented in the form of a deterministic (stochastic) Volterra's series.

**Analysis of Methods of the Deterministic Realization.**

The realization problem for linear dynamic systems was formulated by R.Kalman for the first time in [1-4] it consists in finding of parameters of the linear system according to Output data. This task for linear stationary systems was firstly solved by B. Ho in [1-4]. Other decision for the same task is received in too it was a high time, in [45] where properties of controllability and observe ability of dynamic linear systems were used. Both of these algorithms demand calculations linearly. In [5] Fibonacci`s numbers were described very well.

In [6] R. Kalman stated by following problem.

Fibonacci`s numbers forms minimal rationalization.

Consider problem minimal rationalization. Let we know Map "input – output"

$$y(t) = \int_0^t ce^{A(t-\tau)} BU(\tau)d\tau \,, \tag{1}$$

where

$$W(\bullet) = ce^{A(\bullet)} B \,, \tag{2}$$

$W(\bullet)$ – transitional matrix of dimensional $p \times m$ .

On a transitional matrix (2) it is required to find parameters ( $A, B$ and $C$ ) of following system

$$\dot{x}(t) = Ax(t) + Bu(t) \,, \tag{3}$$

$$y(t) = Cx(t) \,, \tag{4}$$

where $A, B$ and $C$ – matrix, accordingly, $n \times n$ , $n \times m$ and $p \times n$ of dimensional, $u(\bullet) \in R^m$ , $y(\bullet) \in R^p$ , $u(\bullet)$ – input and $y(\bullet)$ – output of a system, $p \le n$ , $m \le n$ .

This problem for the first time was solved Elmer Gilbert on the basis of properties of a controllability and observability, but the first algorithm of realization was given B. Ho, where there was one input and one output, i.e. $m = p = 1$ ( see [1-4].

The algorithm of a solution of a problem of realization consist of the following steps.

*1 step.* On the base of known transitional matrix $W(t)$ (2) we shall calculate block Gankel's matrix $H$ as follows:

$$\begin{aligned} &W(0) = CB = H_1, \\ &\left.\frac{dW(t)}{dt}\right|_{t=0} = CAB = H_2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &\left.\frac{d^{n-1}W(t)}{dt}\right|_{t=0} = CA^{n-1}B = H_n. \end{aligned} \tag{5}$$

*2 step.* Construction Gankel`s matrix $H$

$$H = \begin{pmatrix} H_1 & H_2 & \dots & H_n \\ H_2 & H_3 & \dots & H_{n+1} \\ \dots & \dots & \dots & \dots \\ H_n & H_{n+1} & \dots & H_{2n-1} \end{pmatrix}. \qquad (6)$$

*3 step.* Choose arbitrary matrixes

$$\begin{aligned} P &= (P_1 \quad P_2 \quad \dots \quad P_{n-1}, \quad P_n), \\ M &= (M_1 \quad M_2 \quad \dots \quad M_{n-1}, \quad M_n), \end{aligned} \qquad (7)$$

satisfying to the following matrix`s equation

$$P \cdot H \cdot M = I, \qquad (8)$$

where $I$ – unit matrix and $M_1, M_2, ..., M_{n-1}$ – known matrixes.

*4 step.* From (8) we define $M_n$ by of the following transformations

$$\left( \sum_{i=1}^{n} P_i H_i \quad \sum_{i=1}^{n} P_i H_{i+1} \quad \dots \quad \sum_{i=1}^{n} P_i H_{i+n-1} \right) \begin{pmatrix} M_1 \\ M_2 \\ \dots \\ M_n^* \end{pmatrix} = I$$

$$\left( \sum_{i=1}^{n} P_i H_i \right) M_1 + \left( \sum_{i=1}^{n} P_i H_{i+1} \right) M_2 + \dots + \left( \sum_{i=1}^{n} P_i H_{i+n-2} \right) M_{n-1} + \left( \sum_{i=1}^{n} P_i H_{i+n-1} \right) M_n^* = I$$

$$M_n^* = I - \left[ \sum_{i=1}^{n} P_i H_{i+n-1} \right]^{-1} \left[ \left( \sum_{i=1}^{n} P_i H_i \right) M_1 + \dots + \left( \sum_{i=1}^{n} P_i H_{i+n-2} \right) M_{n-1} \right].$$

*5 step.* Let’s define

$$\vartheta H = \begin{pmatrix} H_2 & H_3 & \dots & H_{n+1} \\ H_3 & H_4 & \dots & H_{n+2} \\ \dots & \dots & \dots & \dots \\ H_{n+1} & H_{n+2} & \dots & H_{2n} \end{pmatrix}. \qquad (9)$$

*6 step.* We define parameters of a system (3)-(4) of the following relations

$$\begin{aligned} A &= P(\vartheta H)M, \\ B &= PH, \\ C &= HM. \end{aligned} \qquad (10)$$

Solution of R.Kalman`s problem.

**R.Kalman`s theorem.** If sequence forms Fibonacci`s numbers, then this Fibonacci`s numbers forms minimal realization accordingly, formulated by R.Kalman.

Because proof of this theorem is very longer. Therefore, we present by following example, which proves of our theorem.

Put we have simple Fibonacci`s numbers

$$0,1,1,2,3,5,8,13,21,34,55,\ldots \tag{11}$$

Put on the base formulas (6-8)

$$H=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix},\; P=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},$$

And PHM=I.
Then

$$PH=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}$$

$$\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix}a_{21} & a_{22}\\ a_{11}+2a_{21} & a_{12}+2a_{22}\end{pmatrix}$$

Father we have

$a_{21}=1,\;\; a_{22}=0,\;\; a_{11}=-2,\;\; a_{12}=1$

$$M=\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}$$

Then because

$$\mathrm{PH}\cdot\mathrm{M}=\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}=I.$$

Accordingly, of formula (10) we received

$$A=\mathrm{P}\cdot\mathrm{IH}\cdot\mathrm{M}=\mathrm{P}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}$$

B=PH=$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}=\begin{pmatrix}0\\ 1\end{pmatrix}$

C=HM=$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}$

In second step we must Fibonacci`s numbers (11)
Because CB=0, then we received first number of Fibonacci`s numbers (11).
Father

CAB=(-1 1) $\begin{pmatrix}0\\ 1\end{pmatrix}=1$

we received second number of Fibonacci`s numbers (11).

$$A^2=C\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}=C\begin{pmatrix}0 & 1\\ -1 & 3\end{pmatrix}$$

$CA^2B=(0\;\;1)\cdot\mathrm{B}=1$

$$A^3=\begin{pmatrix}0 & 1\\ -1 & 3\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-1 & 2\\ -2 & 5\end{pmatrix}$$

$CA^3B=(-1\;\;2)\cdot\begin{pmatrix}0\\ 1\end{pmatrix}=2.$

$A^4=\begin{pmatrix}-1 & 2\\ -2 & 5\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-1 & 3\\ -3 & 8\end{pmatrix}$

$CA^4B=(-1\;\;3)\cdot\begin{pmatrix}0\\ 1\end{pmatrix}=3$

$A^5=\begin{pmatrix}-1 & 3\\ -3 & 8\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 2\end{pmatrix}=\begin{pmatrix}-2 & 5\\ -5 & 13\end{pmatrix}$

$CA^5B=(-2\;\;5)\cdot\begin{pmatrix}0\\ 1\end{pmatrix}=5$

$A^6 = \begin{pmatrix} -2 & 5 \\ -5 & 13 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -3 & 8 \\ -8 & 21 \end{pmatrix}$

$CA^6B = (-3\ \ 8) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 8$

$A^7 = \begin{pmatrix} -3 & 8 \\ -8 & 21 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -5 & 13 \\ -13 & 34 \end{pmatrix}$

$CA^7B = (-5\ \ 13) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 13$

$A^8 = \begin{pmatrix} -5 & 13 \\ -13 & 34 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -8 & 21 \\ -21 & 55 \end{pmatrix}$

$CA^8B = (-8\ \ 21) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 21$

$A^9 = \begin{pmatrix} -8 & 21 \\ -21 & 55 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -13 & 34 \\ -34 & 89 \end{pmatrix}$

$CA^9B = (-13\ \ 34) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 34$

$A^{10} = \begin{pmatrix} -13 & 34 \\ -34 & 89 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -21 & 55 \\ -55 & 144 \end{pmatrix}$

$CA^{10}B = (-21\ \ 55) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 55$

$$A^{15} = \begin{pmatrix} -144 & 377 \\ -377 & 987 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -233 & 610 \\ -610 & 1597 \end{pmatrix}$$

$CA^{15}B = 610$

$$A^{16} = \begin{pmatrix} -233 & 610 \\ -610 & 1597 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -377 & 987 \\ -987 & 2484 \end{pmatrix}$$

$CA^{16}B = 987$

$$H = \begin{bmatrix} h_1\, h_2 & \dots & h_n \\ h_2\, h_3 & \dots & h_{n+1} \\ h_n\ \ h_{n+1} & \cdots & h_{2n+1} \end{bmatrix},$$

$$\Omega H = \begin{bmatrix} h_2\, h_3 & \dots & h_{n+1} \\ h_3\, h_4 & \dots & h_{n+2} \\ h_{n+1}\ \ h_{n+2} & \cdots & h_{2n} \end{bmatrix}.$$

**Conclusion.** In this paper authors considered the R. Kalman`s problem about of Fibonacci numbers. This paper provides an overview of research methods for control theory systems in two concepts “state space” and the “input-output” mapping. In this paper, we consider the problem of R. Kalman on Fibonacci numbers, which consists in the following. Fibonacci numbers formed minimal Realization.

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