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NUMERICAL ALGORITHM FOR SOLVING THE CONTINUATION PROBLEM FOR THE ACOUSTIC EQUATION

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Abstract

In this paper we consider the initial-boundary value problem for the acoustics equation in the temporal-triangular domain. We reduce the original ill-posed problem to an equivalent inverse problem with respect to some direct problem. This direct problem is well-posed. The inverse problem is replaced by a minimization problem. An algorithm for solving the inverse problem by the Landweber iteration method is constructed. We apply the method of successive approximations to the equation, we obtain a natural extension to nonlinear problems. This method leads to optimal convergence rate in certain cases. An analysis of the iterative Landweber method for nonlinear problems depends on the source conditions and additional conditions. Convergence analysis and error estimates are usually made with many assumptions, which are very difficult to verify from a practical point of view. This method leads to optimal convergence rate under certain conditions. Theoretical analysis is confirmed by numerical results. Visual examples are processed numerically.

Keywords: inverse problem, continuation problem, acoustic equation, Landweber method.

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АКУСТИКА ТЕҢДЕУІ ҮШІН ЖАЛҒАСТЫРУ ЕСЕБІНІҢ САНДЫҚ АЛГОРИТМІ

Бұл жұмыста үшбұрышты - уақыт облысында акустика теңдеуі үшін бастапқы шекаралық есеп қарастырылады. Біз бастапқы берілген есепті қандай-да бір тура есепке қатысты кері есепке келтіреміз. Бұл тура есеп қисынды есеп болады. Кері есеп минимизация есебімен ауыстырылады. Жұмыста Ландвебер итерациясы әдісімен кері есепті шешу алгоритмі құрылды. Белгіленген нүктесі бар теңдеуге дәйекті жуықтау әдісін қолданып, сызықтық емес есептің жалғасын аламыз. Бұл әдіс белгілі бір жағдайларда оңтайлы жинақталу жылдамдығына әкеледі. Ландвебердің итеративті әдісін сызықтық емес есептер үшін талдау бастапқы шарттарға және оператордың сызықтық емесдігін шектейтін қосымша шарттарға байланысты. Теориялық талдау сандық нәтижелермен тексеріледі. Көрнекілік үшін мысалдар көрсетілген.

Түйін сөздер: кері есеп, жалғастыру есебі, акустика теңдеуі, Ландвебер әдісі.

Аннотация

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ЧИСЛЕННЫЙ АЛГОРИТМ РЕШЕНИЯ ЗАДАЧИ ПРОДОЛЖЕНИЯ ДЛЯ УРАВНЕНИЯ АКУСТИКИ

В данной статье рассматривается начально - краевая задача для уравнения акустики во временно-треугольной области. Мы сводим исходную некорректную задачу к эквивалентной обратной задаче относительно некоторой прямой задачи. Эта прямая задача является корректной. Обратная задача заменена на задачу минимизации. Построен алгоритм решения обратной задачи, итерационным методом Ландвебера. Применим метод последовательных приближений к уравнению, получим естественное продолжение нелинейных задач. Анализ итерационного метода Ландвебера для нелинейных задач зависит от условий источника и дополнительных условий. Анализ сходимости и оценки ошибок обычно производятся со многими допущениями, которые очень трудно проверить с практической точки зрения. Этот метод приводит к оптимальной скорости сходимости при определенных условиях. Теоретический анализ подтвержден численными результатами. Визуальные примеры обрабатываются численно.

Ключевые слова: обратная задача, задача продолжения, уравнение акустики, метод Ландвебера.

Mathematical modeling of many processes taking place in the real world leads to the study of direct and inverse problems for partial differential equations that have no analogs in classical mathematical physics. In many inverse problems, the desired inhomogeneities are located at a certain depth below the layer of the medium whose parameters are known. In geophysics, this is usually homogeneous or layered media. In this case, an important tool for practitioners are the tasks of continuing geophysical fields from the earth's surface towards the occurrence of heterogeneities.

In acoustics, only small vibrations of the medium are considered, therefore sound waves with small amplitudes are considered. The complete system of general linear acoustic equations for pressure and particle velocity has the form:

$$\begin{cases} \rho \dfrac{\partial v}{\partial t} + \nabla p = 0 \\ \beta \dfrac{\partial p}{\partial t} + \nabla v = 0, \end{cases}$$

where ρ – medium density, v – particle speed, p – wave pressure and β – medium compressibility. Any particular solution to systems of equations is a free wave. We bring the complete system of equations of acoustics to one single equation, with respect to p.

Consider a heterogeneous medium. Let the density of the medium $\rho = \rho(x, y)$ depends on the coordinates. We differentiate the second equation in time and imagine the order of differentiation of speed with respect to time and coordinate replace the value $\dfrac{\partial v}{\partial t}$ on her values from the first equation [1].

$$c^{-2}(x, y)\frac{\partial^2 p}{\partial t^2} = \Delta p - \nabla \ln(\rho(x, y))\nabla p,$$

Where, $c^{-2} = \beta\rho$.

Consider the continuation problem for the acoustics equation in the domain $\Omega = \Delta(L_x) \times (0, L_y)$, where, $\Delta(L_x) = \{(x,t) : x \in (0, L_x), \quad t \in (x, 2L_x - x)\}$ (figure 1).

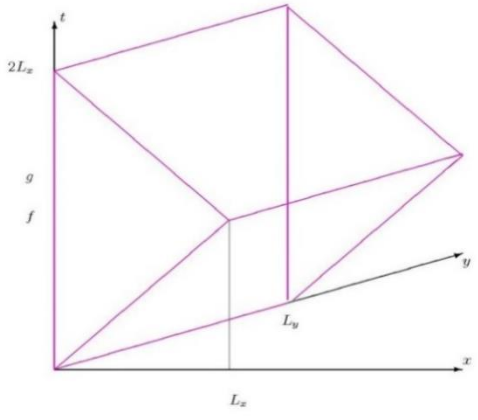

Figure 1. Domain: $\Omega = \Delta(L_x) \times (0, L_y)$

$$u_{tt} = u_{xx} + u_{yy} - \left(\frac{\rho_x}{\rho}u_x + \frac{\rho_y}{\rho}u_y\right) \tag{1}$$

$$u_x(0, y, t) = g(y, t), \tag{2}$$

$$u(0, y, t) = f(y, t), \tag{3}$$

$$u(x,0,t) = u(x,L_y,t) = 0. \tag{4}$$

Problem (1) – (4) is ill-posed [2,3]. Consider the continuation problem (1)– (4) assuming that $\rho(x,y)$ is constant in Ω.

Direct and inverse problems

We consider the ill-posed problem (1) - (4) as the inverse to the next direct problem.

In domain $\Omega = \Delta(L_x) \times (0, L_y)$, here $\Delta(L_x) = \{(x,t): x \in (0, L_x), t \in (x, 2L_x - x)\}$, required to determine $u(x,y,t)$ by given $q(x,y)$ and $g(y,t)$ from the relations:

$$u_{tt} = u_{xx} + u_{yy} - (\frac{\rho_x}{\rho} u_x + \frac{\rho_y}{\rho} u_y), \quad (x,y,t) \in \Delta(L_x), \tag{5}$$

$$u_x(0,y,t) = g(y,t), \quad y \in (0,L_y), t \in (0,2L_x), \tag{6}$$

$$u(x,y,x) = q(x,y), \quad x \in (0,L_x), y \in (0,L_y), \tag{7}$$

$$u(x,0,t) = u(x,L_y,t) = 0, \quad (x,t) \in \Delta(L_x). \tag{8}$$

In the direct problem (5) - (8), it is required to determine $u(x,y,t)$ from given $q(x,y)$ and $g(y,t)$. Direct problem (5) - (8) is well-posed. A more detailed study of this problem can be found in the papers [4-6].

The inverse problem is to determine the function, from relations (5) - (8), according to additional information on solving the direct problem (5) - (8)

$$u(0,y,t) = f(y,t). \tag{9}$$

The numerical solution of such inverse problems leads to a one-dimensional equation of the following problems.

In the domain $\Delta(L_x) = \{(x,t): x \in (0,L_x), t \in (x,2L_x - x)\}$ it is required to determine $v(x,t)$ by given $q(x)$ and $\phi(t)$ from the relations:

$$v_{tt} = v_{xx} - r(x)v(x,t) \in \Delta(L_x) \tag{10}$$

$$v_x(0,t) = \phi(t), t \in (0,2L_x) \tag{11}$$

$$v(x,x) = q(x), x \in (0,L_x), \tag{12}$$

where, $r(x) = \frac{1}{2} \cdot \frac{\rho_{xx}\rho - (\rho_x)^2}{\rho^2} + \frac{1}{4} \cdot (\frac{\rho_x}{\rho})^2$, $\phi(t) = (g(t) - \frac{1}{2} \cdot \frac{\rho_x(0)}{\rho(0)} f(t)) \cdot e^{-\frac{1}{2} ln\rho(0)}$ and $f(t) = f(t) \cdot e^{-\frac{1}{2} ln\rho(0)}$

In the direct problem (10) - (12), it is required to determine $v(x,t)$ by given $q(x)$ and $\phi(t)$.

The inverse problem is to determine the function $q(x)$ from relation (10) - (12) from additional information on solving the direct problem.

$$u(0,t) = f(t). \tag{13}$$

We introduce the operator as follows:

$$A: q(x) \mapsto f(t)$$

$$A: H^1(0,L_x) \mapsto H^1(0,2L_x)$$

We state the inverse problem (10) - (13) in operator form, while leaving all the notation accepted in the work [5].

$$Aq = f. \tag{14}$$

We introduce the objective functional and for solving minimization problem $J(q_n) \to \min$ we apply Landweber iteration.

$$J(q_n) = \| Aq_n - f \|^2 = \int_0^{2L_x} (v(0,t;q_n) - f(t))^2 dt \tag{15}$$

$$q_{n+1} = q_n - \alpha_n J' q_n, \text{ here } \alpha \in \left(0, \|A\|^{-2}\right) \tag{16}$$

We see from Equation (16) that the Landweber algorithm is a special case of gradient descent minimization of a functional $J(q_n)$ [7].

Calculation of the gradient of the objective functional.

Set increment $q_n + \delta q_n$, then

$$\delta v = \tilde{v} - v = v(x,t;q_n + \delta q_n) - v(x,t;q_n). \tag{17}$$

Using the notation (14), we calculate the increment of the objective functional $J(q_n)$.

$$J(q_n+\delta q_n) - J(q_n) = \int_0^{2L_x} [v(0,t;q_n+\delta q_n) - f(t)]^2\,dt - \int_0^{2L_x} [v(0,t;q_n) - f(t)]^2\,dt$$

$$= \int_0^{2L_x} [v(0,t;q_n+\delta q_n) - v(0,t;q_n)]\cdot[v(0,t;q_n+\delta q_n) - f(t) + v(0,t;q_n) - f(t)]dt$$

$$= \int_0^{2L_x} \delta v(0,t;q_n) 2[v(0,t;q_n) - f(t)]dt + o(\|\delta v\|). \tag{18}$$

To obtain the expression $\delta v(0,t;q_n)$, we consider the statement of the perturbed problem for equations (10) - (12).

$$\tilde{v}_{tt} = \tilde{v}_{xx} - r(x)\tilde{v}, \tag{19}$$

$$\tilde{v}_x(0,t) = \phi(t), \tag{20}$$

$$\tilde{v}(x,x) = q_n + \delta q_n. \tag{21}$$

Subtract relations (10) - (12) from relations (19) - (21) and, taking into account (17), we obtain the problem for increment.

$$\delta v_{tt} = \delta v_{xx} - r(x)\delta v, \tag{22}$$

$$\delta v_x(0,t) = 0, \tag{23}$$

$$\delta v(x,x) = \delta q_n. \tag{24}$$

Multiplying (22) by an arbitrary function $\psi(x,t)$ and we integrate over $\Delta(L_x)$.

$$0 = \iint_{\Delta(L_x)} (\delta v_{tt} - \delta v_{xx} + r(x)\delta v)\psi dxdt = \int_0^{L_x}\int_x^{2L_x - x} \psi\delta v_{tt} dtdx - \int_0^{L_x}\int_0^t \psi\delta v_{xx} dxdt$$

$$- \int_{L_x}^{2L_x}\int_0^{2L_x - t} \psi\delta v_{xx} dxdt + \iint_{\Delta(L_x)} r(x)\psi\delta v dxdt.$$

Integrate by parts this expression

$$\int_0^{L_x} [(\psi\delta v_t)(x,2L_x - x) - \underline{(\psi\delta v_t)(x,x)} - (\psi_t\delta v)(x,2L_x - x) + \underline{\underline{(\psi_t\delta v)(x,x)}} + \int_x^{2L_x - x} \psi_{tt}\delta v dt]dx$$

$$- \int_0^{L_x} [\underline{(\psi\delta v_x)(t,t)} - (\psi \overset{0}{\delta v_x})(0,t) - \underline{\underline{(\psi_x\delta v)(t,t)}} + (\psi_x\delta v)(0,t) + \int_0^t \psi_{xx}\delta v dx]dt$$

$$- \int_{L_x}^{2L_x} [(\psi\delta v_x)(2L_x - t,t) - (\psi \overset{0}{\delta v_x})(0,t) - (\psi_x\delta v)(2L_x - t,t) + (\psi_x\delta v)(0,t) + \int_0^{2L_x - t} \psi_{xx}\delta v dx]dt$$

$$+ \iint_{\Delta(L_x)} r(x)\psi\delta v dxdt.$$

Given (23) and due to the fact that

$\psi_x(x,2L_x - x) - \psi_t(x,2L_x - x) = \frac{d\psi}{dx}\Big|_{\frac{dt}{dx}=-1} = \psi_t(x,2L_x - x)$ (directional derivative $t = 2L_x - x$);

$\delta v_x(x,2L_x - x) - \delta v_t(x,2L_x - x) = \frac{d\delta v}{dx}\Big|_{\frac{dt}{dx}=-1} = \delta v_t(x,2L_x - x)$ (directional derivative $t = 2L_x - x$);

$\psi_x(x,x)+\psi_t(x,x)=\frac{d\psi}{dx}|_{\frac{dt}{dx}=1}=\psi_t(x,x)$ (directional derivative $x=t$);

$\delta v_x(x,x)+\delta v_t(x,x)=\frac{d\delta v}{dx}|_{\frac{dt}{dx}=1}=(\delta q)_t(x)$ (directional derivative $t=x$), integrate by parts and get

$$0=\iint_{\Delta(L_x)}(\psi_{tt}-\psi_{xx}+r(x)\psi)\delta v dxdt$$

$$+\int_0^{L_x}[\psi(x,2L_x-x)(\delta v(x,2L_x-x))_t|_{t=2L_x-x}-\delta v(x,2L_x-x)(\psi(x,2L_x-x))_t|_{t=2L_x-x}]dx$$

$$+\int_0^{L_x}[\delta v(x,x)(\psi(x,x))_t|_{t=x}-\psi(x,x)(\delta v(x,x))_t|_{t=x}]dx-\int_0^{2L_x}\psi_x(0,t)\delta v(0,t)dt$$

Where does the formulation of the conjugate problem follow

$$\psi_{tt}=\psi_{xx}-r(x)\psi, \tag{25}$$

$$\psi(x,2L_x-x)=0, \tag{26}$$

$$\psi_x(0,t)=2(u(0,t)-f(t)). \tag{27}$$

Then, given (18), we obtain

$$\langle\delta q_n, J'q_n\rangle=\int_0^{L_y}\int_0^{L_x}\delta q(2\psi(x,x))_t|_{t=x}\,dx.$$

By definition the main part of the functional increment is the gradient [3], i.e.

$$J'q_n=(2\psi(x,x))_t|_{t=x}, \tag{28}$$

here $\psi(x,t)$ is the solution of the conjugate problem (25) – (27).

An algorithm for solving the inverse problem by the Landweber iteration method

1) Set descent parameter α.
2) We choose the initial approximation q_0.
3) We solve the direct problem (10)-(12) with a given q_n.
4) We calculate the value of the functional by the formula (15).
5) If the value of the objective functional is not small enough, then we solve the conjugate problem (25) - (27).
6) We calculate the gradient of the functional by the formula (28).
7) We calculate the following approximation $q_{n+1}=q_n-\alpha J'q_n$, here $\alpha\in\left(0;\|A\|^{-2}\right)$ and turn to step 2.

Numerical results

For the numerical solution of the inverse problem, we use the method inverse transformation difference schemes [8, 9].

Consider the finite-difference scheme of this problem.

In domain Ω we construct a grid ω_h with step $h=l/N$, where N is a positive integer.

Then in the grid we write the corresponding difference direct problem for the equations of acoustics

Then in the grid $\omega_h=\{x=ih, t=kh; i=\overline{1,N-1}, k=\overline{i,2N-i}\}$ we write the corresponding difference form of direct problem for the acoustics equation. Thus, the problem (10) - (12) has the following form:

Thus, the problem (10) - (12) has the following form:

$$u_i^{k+1}+u_i^{k-1}=u_{i+1}^k+u_{i-1}^k-h^2\cdot\frac{r_{i+1}+r_{i-1}}{4}u_{i+1}^k-h^2\cdot\frac{r_{i+1}+r_{i-1}}{4}u_{i-1}^k$$

$$u_1^k=\frac{1}{2}(u_0^{k+1}+u_0^{k-1})+h\cdot\phi^k$$

$$u_i^i=q_i,$$

$$u_0^k = f_k.$$

For convenience, we introduce new notation: $a_{i+1} = 1 - h^2 \cdot \frac{r_{i+1} + r_{i-1}}{4}$, then we get,

$$u_i^{k+1} + u_i^{k-1} = a_{i+1} u_{i+1}^k + a_{i+1} u_{i-1}^k \quad (29)$$

$$u_1^k = \frac{1}{2}(u_0^{k+1} + u_0^{k-1}) + h \cdot \phi^k \quad (30)$$

$$u_i^i = q_i, \quad (31)$$

$$u_0^k = f_k. \quad (32)$$

Taking into account (31) and setting $k = i+1$ in (29) we obtain the following expression

$$q_{i+1} = q_i + u_i^{i+2} - u_{i-1}^{i+1} \quad (33)$$

Algorithm for the method of inverse transform difference schemes

1) Calculated by the formula (32) $q_0 = u_0^0 = f_0$.

2) Calculated by the formula (31) $q_1 = \frac{1}{2}(u_0^2 + u_0^0) + h \cdot \phi^1$ где $u_0^2 = f_2$.

3) Calculated by the formula (32) $u_0^4 = f_4$; by formula (30) u_1^5; by formula (33) q_2.

4) And so on, we calculate by the formula (32) $u_0^{2i} = f_{2i}$; by formula (30) u_1^{2i-1}; by formula (29) u_{i+1}^k along the characteristic; calculated by the formula (33) q_i.

Computational Experiment

Let's conduct a computational experiment for $N = 100$, $h = 0.01$, $g(t) = 0$, $\rho(x) = e^{-\frac{(x-0.5)^2}{2b^2}}$, $b = 0.1$, $q(x) = 0.3(e^{-\frac{(x-0.5)^2}{2b^2}} + e^{-\frac{(x-0.75)^2}{2b^2}})$, (figure 2, figure 3).

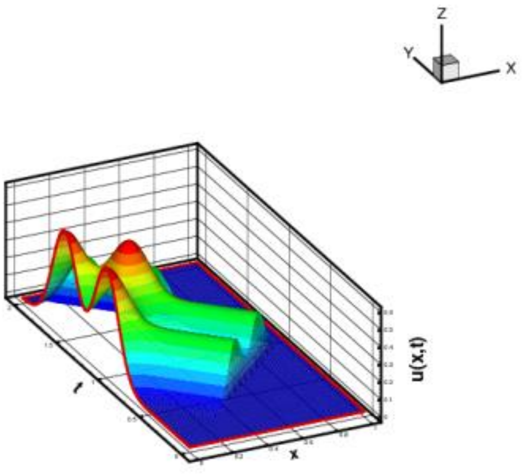

Figure 2. u(x,t)

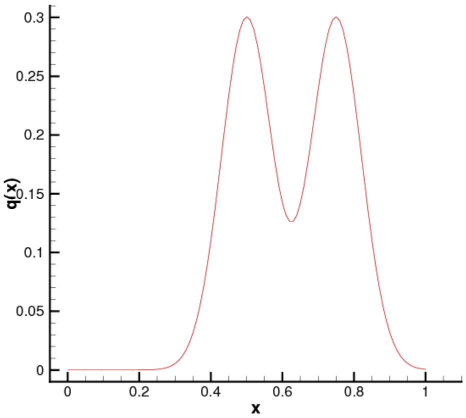

Figure 3. q(x)

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