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BOUNDARY VALUE PROBLEM FOR A LOADED HEAT CONDUCTIVITY OPERATOR

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Abstract

The steadily growing interest in the study of loaded differential equations is explained by the expanding scope of their applications and the fact that loaded equations constitute a special class of functional differential equations with their own specific tasks. These equations are used in the study of inverse problems of differential equations, which have important applied significance. The paper investigates the solvability problems of homogeneous and nonhomogeneous boundary value problems, as well as spectral issues for loaded differential operators of mathematical physics, when the loaded terms are not a weak perturbation of the differential part of the operator. They require special theoretical research.

Keywords: boundary value problem, unlimited domain, differential operator, spectrally loaded equations, spectrally loaded heat equations, generalized spectral problems, characteristic integral equations.

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ЖҮКТЕЛГЕН ЖЫЛУӨТКІЗГІШТІК ОПЕРАТОР ҮШІН ШЕКТІК ЕСЕП

Жүктелген дифференциалдық теңдеулерді зерттеуге деген тұрақты өсіп келе жатқан қызығушылық олардың қолдану аясының кеңеюімен және жүктелген теңдеулердің өзіндік нақты міндеттері бар функционалдық дифференциалдық теңдеулердің арнайы класын құрайтындығымен түсіндіріледі. Бұл теңдеулер маңызды қолданбалы мәні бар дифференциалдық теңдеулердің кері есептерін зерттеуде қолданылады. Жұмыста біртекті және біртекті емес шекаралық есептердің шешілу есептері, сондай-ақ жүктелген терминдер оператордың дифференциалдық бөлігінің әлсіз бұзылуы болмаған кезде математикалық физиканың жүктелген дифференциалдық операторлары үшін спектрлік мәселелер зерттеледі. Олар арнайы теориялық зерттеулерді қажет етеді.

Түйін сөздер: шектік есептер, шексіз аймақ, дифференциалдық оператор, спектралды жүктелген теңдеулер, спектралды жүктелген жылу теңдеулері, жалпыланған спектрлік есептер, сипатталған интегралдық теңдеулер.

Аннотация
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КРАЕВАЯ ЗАДАЧА ДЛЯ НАГРУЖЕННОГО ОПЕРАТОРА ТЕПЛОПРОВОДНОСТИ

Неуклонно растущий интерес к изучению нагруженных дифференциальных уравнений объясняется расширением области их приложений и тем, что нагруженные уравнения составляют особый класс функционально-дифференциальных уравнений со своими специфическими задачами. Эти уравнения используются при изучении обратных задач дифференциальных уравнений, имеющих важное прикладное значение. В работе исследуются проблемы разрешимости однородных и неоднородных краевых задач, а также спектральные вопросы для нагруженных дифференциальных операторов математической физики, когда нагруженные члены не являются слабым возмущением дифференциальной части оператора. Они требуют специального теоретического исследования.

Ключевые слова: краевая задача, неограниченная область, дифференциальный оператор, спектрально нагруженные уравнения, спектрально нагруженные уравнения теплопроводности, обобщенные спектральные задачи, характеристические интегральные уравнения.

1. Introduction

Loaded differential equations as an object of the theory of partial differential equations and mathematical modeling appeared quite a long time ago. Loaded differential equations find numerous applications in practical problems. For example, the problem of the longitudinal movement of a load suspended from an elastic thread, the problem of vibrations of a string loaded with concentrated masses, the study of torsional vibrations of a thread to the end of which a mass is suspended, the problem of long-term forecasting and regulation of the level of groundwater and soil moisture. Boundary value problems for loaded equations have become particularly relevant in connection with the study of the vibration stability of aircraft wings, since to solve such a problem it is necessary to calculate the natural frequencies of a wing loaded with motors. In addition, similar problems occur when calculating the natural oscillations of antennas loaded with concentrated capacitances and self-inductions. Loaded equations also arise in the study of nonlinear equations, particle transport equations, optimal control problems, in the numerical solution of integro-differential equations, in the equivalent transformation of non-local boundary value problems, etc.

Loaded differential equations represent a special class in connection with specific problems. Equations of this kind arise in the study of nonlinear equations, for example particle transport equations, optimal control problems, inverse problems, in the case of an equivalent transformation of nonlocal problems, etc. [1].

Interest in the study of regional value problems for loaded differential equations is associated with the growing volume of their applications [2-5].

In this paper, we study boundary value problems for spectrally loaded parabolic equations in unlimited domains, when the order of the derivative of the loaded term coincides with the order of the differential part of the equation and the load point in the spatial variable moves with variable speed. In this case, new properties of the loaded differential operator appear, which are not inherent in operators with a weak perturbation. They require special theoretical research [6-7].

This serves to substantiate both the theoretical and practical relevance of the formulation and study of boundary value problems for spectrally loaded differential equations for the study of the spectral characteristics of these problems, as well as the establishment of criteria for the correctness of boundary value problems and the selection of the corresponding functional classes.

In this paper, boundary value problems for spectrally loaded parabolic equations are investigated, where the order of the derivatives in the loaded terms of the differential equation is equal to the order of the derivatives included in the equation itself. In the study of boundary value problems, the method of reduction of the boundary value problem to singular Volterra integral equations of the second-order is used [8-10].

2. Problem statements

Consider in the domain Q = $\{x \in R_+, t \in R_+\}$ boundary value problems for spectrally loaded heat equation:

$$L_\lambda u = f \Leftrightarrow \begin{cases} u_t - u_{xx} + \lambda u_{xx}(x,t)\big|_{x=\alpha(t)} = f, \\ u(x,0) = 0, u(0,t) = 0; \end{cases} \tag{1}$$

$$L_\lambda^* v = g \Leftrightarrow \begin{cases} -v_t - v_{xx} + \bar{\lambda}\,\delta''(x-\alpha(t)) \otimes \int\limits_0^\infty v(\xi,t)d\xi = g, \\ v(x,\infty) = 0, v(0,t) = v(\infty,t) = v_x(\infty,t) = 0; \end{cases} \tag{2}$$

and generalized spectral problems:

$$L_1 u = -\lambda u_{xx}(x,t)_{|x=\alpha(t)} \Leftrightarrow \begin{cases} u_t - u_{xx} = -\lambda u_{xx}(x,t)_{|x=\alpha(t)}, \\ u(x,0) = 0, u(0,t) = 0; \end{cases} \tag{3}$$

$$L_1^* v = -\bar{\lambda} \cdot \delta''(x-\alpha(t)) \otimes \int\limits_0^\infty v(\xi,t)d\xi \Leftrightarrow \begin{cases} -v_t - v_{xx} = -\bar{\lambda} \cdot \delta''(x-\alpha(t)) \otimes \int\limits_0^\infty v(\xi,t)d\xi, \\ v(x,\infty) = 0, v(0,t) = v(\infty,t) = v_x(\infty,t) = 0; \end{cases} \tag{4}$$

The specified functions are selected from the classes

$$\left(\alpha(t)\right)^{\frac{\omega-3/2}{\omega}} f \in L_1(Q),\ \left(\alpha(t)\right)^{-1}(x+\sqrt{t})\, g \in L_\infty(Q)),$$

$$\left(\alpha(t)\right)^{\frac{\omega-3/2}{\omega}} \left(\frac{\partial^2}{\partial x^2} \int_0^t \int_0^\infty G(x,\xi,t-\tau) f(\xi,\tau) d\xi d\tau \right)_{|x=\alpha(t)} \in L_1(R_+)\,. \tag{5}$$

$\delta(x-t) \in \mathrm{E}'(\mathrm{Q})$ is the delta function centered on an open line $\mathrm{x} = \alpha(\mathrm{t})$ of the domain Q, $\mathrm{E}'(\mathrm{Q})$ is a space of generalized functions with compact support in the domain Q,

$$erfz \frac{2}{\sqrt{\pi}} \int_0^a \exp(-\xi^2) d\xi\,,$$

Here the Green function $G(x,\xi,t-\tau)$ is defined by formula

$$G(x,\xi,t-\tau) = \frac{1}{2\sqrt{\pi t}} \left\{ \exp\left(-\frac{(x-\xi)^2}{4t} \right) - \exp\left(-\frac{(x+\xi)^2}{4t} \right) \right\}. \tag{6}$$

Remark 1.

If the function f does not depend on the variable *x*, then the second condition for the function f from (5) follows from the first: $\left(\alpha(t)\right)^{\frac{\omega-3/2}{\omega}} f \in L_1(Q)$.

In problems (1) - (4), it is assumed that the motion of the load point is described by the function $x(t) = \alpha(t)$ under condition $\lim\limits_{t\to 0} \frac{\alpha(t)}{\sqrt{t}} = \infty$.

The functional classes U and V for solving boundary value problems and the domains of the operators L and L*, D(L) and D(L*) are defined as follows, respectively

$$U = \left\{ u \middle| \alpha(t)(x+\sqrt{t})^{-1} u, \left(\alpha(t)\right)^{\frac{\omega-3/2}{\omega}} (u_t - u_{xx}) \in L_1(Q), \left(\alpha(t)\right)^{\frac{\omega-3/2}{\omega}} u_{xx}(x,t)_{|x-\alpha(t)} \in L_1(R_+), \tag{7}$$

$$V = \left\{ v \middle| \left(\alpha(t)\right)^{\frac{3/2-\omega}{\omega}} v,\ \left(\alpha(t)\right)^{-1}(x+\sqrt{t})(v_t + v_{xx}) \in L_\infty(Q),\ \left(\alpha(t)\right)^{\frac{3/2-\omega}{\omega}} \int_0^\infty v(\xi,t) d\xi \in L_\infty(R_+) \right\}, \tag{8}$$

$$D(L_\lambda) \equiv D(L_1) = \left\{ u \middle| u \in U, u(x,0) = 0, u(0,t) = 0 \right\}, \tag{9}$$

$$D(L_\lambda^*) \equiv D(L_1^*) = \left\{ v \middle| v \in V, v(x,\infty) = 0, v(0,t) = 0, v(\infty,t) = 0, v_x(\infty,t) = 0 \right\}. \tag{10}$$

The boundary value problem (2) is conjugate to problem (1). Indeed, according to (1)-(10) we have

$$< Lu, v >=< u, L^* v > \ \forall u \in D(L)\,, \forall v \in D(L^*)\,.$$

Problem 1. It is necessary to investigate the solvability of boundary value problems (1) and (2) under conditions (5)-(10).

Problem 2. It is necessary to study the spectral problems (3) and (4) for determining pairs $\{\lambda, u_\lambda(x,t)\}$ and $\{\lambda, v_\lambda(x,t)\}$ under conditions (7)-(10).

3. The reduction of boundary value problems to integral equations

We reduce boundary value problems (1) and (2) to the study of adjoint singular Volterra integral equations of the second kind. For this, we invert the differential part in the boundary value problem (1) and have:

$$u(x,t) = -\lambda\int_0^t erf\left(\frac{x}{2\sqrt{t-\tau}}\right)u_{\eta\eta}(\eta,\tau)\Big|_{\eta=\alpha(\tau)}d\tau + \int_0^t\int_0^\infty G(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau\,, \tag{11}$$

where

$$K_0(x,t-\tau) = \int_0^\infty G(x,\xi,t-\tau)d\xi = erf\left(\frac{x}{2\sqrt{t-\tau}}\right).$$

It follows from relation (11) that to find a solution to problem (1), it is sufficient to determine the loaded summand $u_{xx}(x,t)_{|x=\alpha(t)}$. To do this, we differentiate both parts of the relation (11) by the variable x twice and enter the following notation

$$\mu(t) = (\alpha(t))^{\frac{\omega-3/2}{\omega}}u_{xx}(x,t)_{|x=\alpha(t)},$$

$$K_2(t,\tau) = \left[\frac{\alpha(t)}{\alpha(\tau)}\right]^{\frac{\omega-3/2}{\omega}}\cdot\frac{\alpha(t)}{2\sqrt{\pi}(t-\tau)^{3/2}}\exp\left(-\frac{(\alpha(t))^2}{4(t-\tau)}\right),$$

$$f_1(t) = (\alpha(t))^{\frac{\omega-3/2}{\omega}}\left(\frac{\partial^2}{\partial x^2}\int_0^t\int_0^\infty G(x,\xi,t-\tau)f(\xi,\tau)d\xi d\tau\right)_{|x=\alpha(t)}, \tag{12}$$

we obtain the integral equation

$$\mathrm{K}_{2\lambda}\mu \equiv (I-\lambda\mathrm{K}_2)\mu \equiv \mu(t) - \lambda\int_0^t K_2(t,\tau)\mu(\tau)d\tau = f_1(t),\ t\in R_+\,. \tag{13}$$

Remark 2. Note that the boundedness of function $f_1(t)$ (12) on R+ follows from condition (5) on f.

The kernel $K_2(t,\tau)$ of the integral equation has the following properties:

1^0 the kernel $K_2(t,\tau)$ continuously, $0<\tau<t<\infty$;

2^0 the kernel $K_2(t,\tau)$, $0<\tau<t<\infty$;

3^0 the following limiting ratio is valid

$$\lim_{t\to+0}\int_0^t K_2(t,\tau)d\tau = 0\,. \tag{14}$$

Proof of property 3^0.

$$\lim_{t\to+0}\int_0^t K_2(t,\tau)d\tau = \lim_{t\to+0}\int_0^t\left[\frac{\alpha(t)}{\alpha(\tau)}\right]^{\frac{\omega-3/2}{\omega}}\frac{\alpha(t)}{2\sqrt{\pi}(t-\tau)^{3/2}}exp\left(-\frac{(\alpha(t))^2}{4(t-\tau)}\right)d\tau =$$

$$= \lim_{t\to+0}\int_0^t\left[\frac{\alpha(\tau)}{\alpha(t)}\right]^{\frac{3/2-\omega}{\omega}}\frac{4}{\sqrt{\pi}(\alpha(t))^2}\cdot\left(\frac{\alpha(t)}{2\sqrt{t-\tau}}\right)^3\exp\left(-\frac{(\alpha(t))^2}{4(t-\tau)}\right)d\tau \le$$

$$\le \lim_{t\to+0}\frac{4}{\sqrt{\pi}}\cdot\left(\frac{3}{2}\right)^{\frac{3}{2}}\cdot e^{-\frac{3}{2}}\cdot\frac{1}{[t(1+\alpha_0(t))]^{2\omega}}\int_0^t d\tau \le \lim_{t\to+0}\frac{4}{\sqrt{\pi}}\left(\frac{3}{2}\right)^{\frac{3}{2}}e^{-\frac{3}{2}}t^{1-2\omega} = 0$$

Conditions were used here: $0<\tau<t<\infty$, $\frac{3}{2}-\omega>1$ and inequality

$z^m \exp(-z^n) \le \left(\frac{m}{n}\right)^{\frac{m}{n}} \exp\left(-\frac{m}{n}\right)$, for any z>0, and also that $\alpha(t) = [t(1+\alpha_0(t))]^{\omega}$ and $1+\alpha_0(t) \le C$.

*Remark 3***.** From the relation (14) it follows that the integral equation (13) refers to "Volterra's for which the solution exists and is unique. We proceed to the consideration of the conjugate boundary value problem (2) and inverting its differential part, we obtain:

$$v(x,t) = -\bar{\lambda}\int_t^{\infty}\int_0^{\infty} G(x,\xi,\tau-t)\delta''(\xi-\alpha(\tau)) \otimes \int_0^{\infty} v(\eta,\tau)d\eta d\xi d\tau +$$

$$+\int_t^{\infty}\int_0^{\infty} G(x,\xi,\tau-t)g(\xi,\tau)d\xi d\tau ,$$

Integrating relation (19) over the variable x from 0 to ∞ and denoting

$$v(t) = (\alpha(t))^{\frac{3/2-\omega}{\omega}} \int_0^{\infty} v(\eta,t)d\eta,$$

we obtain the integral equation

$$\mathrm{K}_{2\lambda}^{*} v \equiv \left(I - \bar{\lambda}\mathrm{K}_2^{*}\right)v \equiv \upsilon(t) - \bar{\lambda}\int_t^{\infty} K_2(\tau,t)\upsilon(\tau)d\tau = g_1(t), \quad t \in R_+, \qquad (15)$$

where the following notation was used:

$$K_2(\tau,t) = \left(\frac{\alpha(t)}{\alpha(\tau)}\right)^{\frac{3/2-\omega}{\omega}} \frac{\alpha(\tau)}{2\sqrt{\pi}(\tau-t)^{3/2}} \exp\left(-\frac{(\alpha(\tau))^2}{4(\tau-t)}\right),$$

$$g_1(t) = (\alpha(t))^{\frac{3/2.-\omega}{\omega}} \int_t^{\infty}\int_0^{\infty} erf\left(\frac{\xi}{2\sqrt{\tau-t}}\right) g(\xi,\tau)d\xi d\tau. \qquad (16)$$

Remark 4. Note that the integrability of function (16) $g_1(t)$ on R_+ follows from condition (5) on g .

Note that the kernel of the conjugate integral equation (15) has the following property:

$$\lim_{t\to\infty}\int_t^{\infty} K_2(\tau,t)d\tau = 1. \qquad (17)$$

It follows from the limit relation (17) that the norm of an integral operator acting in the space of bounded and continuous functions and defined by the kernel $K_{2\lambda}^{*}$ is equal to one (although the kernel $K_{2\lambda}^{*}$ has an integrable singularity). This fundamentally distinguishes equation (15) from Volterra equations of the second kind. Thus, the solution of conjugate boundary value problems (1), (2) is reduced to the study of a pair of conjugate integral equations (13) and (15), which in the future we will call the initial ones.

4. Study of characteristic integral equations

Consider the characteristic integral equations corresponding to integral equations (13) and (15):

$$\mathrm{K}_{\lambda}\mu \equiv (I - \lambda\mathrm{K})\mu \equiv \mu(t) - \lambda\int_0^t K(t,\tau)\mu(\tau)d\tau = f_1(t), \; t \in R_+,$$

$$\mathrm{K}_{\lambda}^{*} v \equiv \left(I - \bar{\lambda}\mathrm{K}^{*}\right)v \equiv \upsilon(t) - \bar{\lambda}\int_t^{\infty} K(\tau,t)\upsilon(\tau)d\tau = g_1(t), \quad t \in R_+, \qquad (18)$$

where

$$K(\tau,t) = \left(\frac{\alpha(t)}{\alpha(\tau)}\right)^{\frac{3/2-\omega}{\omega}} \cdot \frac{(1-2\omega)^{3/2}[\alpha(\tau)]^{-2}\left([\alpha(\tau)]^{\frac{1}{\omega}}\right)'}{2\sqrt{\pi}\left([\alpha(\tau)]^{\frac{1-2\omega}{\omega}} - [\alpha(t)]^{\frac{1-2\omega}{\omega}}\right)^{3/2}} \times$$

$$\times exp\left(-\frac{1-2\omega}{4\left(\left[\alpha(t)\right]^{\frac{1-2\omega}{\omega}}-\left[\alpha(\tau)\right]^{\frac{1-2\omega}{\omega}}\right)}\right). \tag{19}$$

The kernel of the characteristic equation $K(\tau,t)$ has the same properties as the kernel $K_2(\tau,t)$ and the following limiting relation is valid for it:

$$\lim_{t\to\infty}\int_t^{\infty} K(\tau,t)d\tau = 1, \tag{20}$$

It can be shown that the difference between the kernels $\widetilde{K}(\tau,t) = K_2(\tau,t) - K(\tau,t)$ must have a weak feature [12-15]. This follows from the statement of the next theorem.

*Theorem 1***.** If the function $\alpha(t) = \left[t(1+\alpha_0(t))\right]^{\omega}$, where $\alpha_0(t) = t^{\beta}\sigma(t), \beta > 0$, the function $\sigma(t)$ is twice continuously differentiable for $0 < t < \tau < \infty$ $|\sigma(t)| \le C$, $\sigma(t) \ne 0$, then the following estimate is valid

$$|K_2(\tau,t) - K(\tau,t)| \le C(\omega)\left[\frac{\alpha(\tau)}{\alpha(t)}\right]^{-\frac{\omega-3/2}{\omega}}\frac{t^{1/2+\omega}}{\tau^{3/2}\sqrt{\tau-t}}\times$$

$$\times\left[\exp\left((2\omega-1)\frac{\left[\alpha(t)\right]^{\frac{2\omega-1}{\omega}}\left[\alpha(\tau)\right]^{\frac{2\omega-1}{\omega}}}{8\left(\left[\alpha(t)\right]^{\frac{2\omega-1}{\omega}}-\left[\alpha(\tau)\right]^{\frac{2\omega-1}{\omega}}\right)}\right)+\exp\left(\frac{(\alpha(t))^2}{8(t-\tau)}\right)\right].$$

To prove this theorem, we reduce integral equations (15) and (18) to equations on a finite interval $(0,t)$. To do this, we make changes of variables in these equations:

$$\alpha(t) = \frac{1}{\alpha(t_1)}, \quad \alpha(\tau) = \frac{1}{\alpha(\tau_1)}.$$

Then integral equations (15) and (18) take the form, respectively:

$$\nu(t_1) - \bar{\lambda}\int_0^{t_1}\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{\tau_1^{-1/2}t_1^{3/2}\left[\alpha(\tau_1)\right]^{-1}}{2\sqrt{\pi}(t_1-\tau_1)^{3/2}}\exp\left(-\frac{t_1\tau_1\left[\alpha(\tau_1)\right]^{-2}}{4(t_1-\tau_1)}\right)\nu(\tau_1)d\tau_1 = g_1(t_1),$$

$$\nu(t_1) - \bar{\lambda}\int_0^{t_1}\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{(1-2\omega)^{3/2}}{2\sqrt{\pi}}\cdot\frac{\left[\alpha(t_1)\right]^{\frac{3/2(1-2\omega)}{\omega}}\left[\alpha(\tau_1)\right]^{-\frac{1+2\omega}{2\omega}}\left[\alpha(\tau_1)^{\frac{1}{\omega}}\right]'}{\left(\left[\alpha(t_1)\right]^{\frac{1-2\omega}{\omega}}-\left[\alpha(\tau_1)\right]^{\frac{1-2\omega}{\omega}}\right)^{3/2}}\times$$

$$\times exp\left(-\frac{(1-2\omega)\left[\alpha(t_1)\right]^{\frac{1-2\omega}{\omega}}\left[\alpha(\tau_1)\right]^{\frac{1-2\omega}{\omega}}}{4\left(\left[\alpha(t_1)\right]^{\frac{1-2\omega}{\omega}}-\left[\alpha(\tau_1)\right]^{\frac{1-2\omega}{\omega}}\right)}\right)\nu(\tau_1)d\tau_1 = g_1(t_1).$$

We denote these kernels of equations (19) and (20) by $K_2'(t_1,\tau_1)$ and $K'(t_1,\tau_1)$ and write them in the form:

$$K_2'(t_1,\tau_1)=P_2'(t_1,\tau_1)e^{-Q_2'(t_1,\tau_1)},\ K'(t_1,\tau_1)=P'(t_1,\tau_1)e^{-Q'(t_1,\tau_1)},$$

where

$$P_2'(t_1,\tau_1)=\left(\frac{\alpha(\tau_1)}{\alpha(t_1)}\right)^{\frac{3/2-\omega}{\omega}}\frac{\tau_1^{-1/2}t_1^{3/2}[\alpha(\tau_1)]^{-1}}{2\sqrt{\pi}(t_1-\tau_1)^{3/2}},\qquad Q_2'(t_1,\tau_1)=\frac{t_1\tau_1[\alpha(\tau_1)]^{-2}}{4(t_1-\tau_1)},$$

$$P'(t_1,\tau_1)=\left(\frac{\alpha(\tau_1)}{\alpha(t_1)}\right)^{\frac{3/2-\omega}{\omega}}\frac{(1-2\omega)^{3/2}}{2\sqrt{\pi}}\frac{[\alpha(t_1)]^{\frac{3/2(1-2\omega)}{\omega}}[\alpha(\tau_1)]^{-\frac{1+2\omega}{2\omega}}\left[\alpha(\tau_1)^{\frac{1}{\omega}}\right]'}{\left([\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}\right)^{3/2}},$$

$$Q'(t_1,\tau_1)=\frac{(1-2\omega)[\alpha(t_1)]^{\frac{1-2\omega}{\omega}}[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}}{4\left([\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}\right)}.$$

The following theorem holdsю

Theorem 2. If the function $\alpha(t_1)=[t_1(1+\alpha_0(t_1))]^{\omega}$, where $\alpha_0(t_1)=t_1^{\beta}\sigma(t_1),\beta>0$, and the function $\sigma(t_1)$ is twice continuously differentiable for $0<\tau_1<t_1<\infty$, *u* $|\sigma(t_1)|\le C$, $\sigma(t_1)\neq 0$, then the following estimate holds:

$$|K'(t_1,\tau_1)-K_2'(t_1,\tau_1)|\le C(\omega)\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{t_1^{-\omega}}{\sqrt{t_1-\tau_1}}\times$$

$$\times\left[\exp\{-Q'(t_1,\tau_1)/2\}+\exp\{-Q_2'(t_1,\tau_1)/2\}\right]. \tag{21}$$

First we prove a few lemmas.

Lemma 1. If the function $\alpha(t_1)=[t_1(1+\alpha_0(t_1))]^{\omega}$, where $\alpha_0(t_1)=t_1^{\beta}\sigma(t_1),\beta>0$ and $\alpha_0(t_1)$ increases monotonically at $0<\tau_1<t_1<\infty$, $|\sigma(t_1)|\le C$, then the following estimate holds:

$$|P'(t_1,\tau_1)-P_2'(t_1,\tau_1)|\le\overline{M}\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\cdot\frac{t_1^{1-\omega+\beta}}{(t_1-\tau_1)^{3/2}}.$$

Proof of Lemma 1.

$$|P'(t_1,\tau_1)-P_2'(t_1,\tau_1)|=\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\left|\frac{1}{2\sqrt{\pi}}\cdot\frac{(1-2\omega)^{3/2}[\alpha(t_1)]^{\frac{3}{2}\cdot\frac{1-2\omega}{\omega}}[\alpha(\tau_1)]^{\frac{-1-2\omega}{2\omega}}\left([\alpha(\tau_1)]^{\frac{1}{\omega}}\right)'}{\left([\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}\right)^{3/2}}-\right.$$

$$\left.-\frac{1}{2\sqrt{\pi}}\cdot\frac{t_1^{3/2}\tau_1^{-1/2}[\alpha(\tau_1)]^{-1}}{(t_1-\tau_1)^{3/2}}\right|=\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{1}{2\sqrt{\pi}}\cdot\left([\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}\right)^{-3/2}\times$$

$$\times\left|(1-2\omega)^{3/2}[\alpha(t_1)]^{\frac{3}{2}\cdot\frac{1-2\omega}{\omega}}[\alpha(\tau_1)]^{\frac{-1-2\omega}{2\omega}}([\alpha(\tau_1)]^{\frac{1}{\omega}})'-t_1^{3/2}\tau_1^{-1/2}[\alpha(\tau_1)]^{-1}\cdot\left(\frac{[\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}}{t_1-\tau_1}\right)^{3/2}\right|=$$

$$=\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{1}{2\sqrt{\pi}}\cdot\frac{1}{\left([\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}\right)^{3/2}}\left|(1-2\omega)^{3/2}[t_1(1+\alpha_0(t_1))]^{\frac{3-6\omega}{2}}[\tau_1(1+\alpha_0(\tau_1))]^{\frac{-1-2\omega}{2}}\times\right.$$

$$\left.\times\left(1+\alpha_0(\tau_1)+\tau_1\alpha_0'(\tau_1)\right)-t_1^{3/2}\tau_1^{-1/2}[\tau_1(1+\alpha_0(\tau_1))]^{-\omega}\left\{([\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}})'_{|\tau_1=t_1}-\frac{1}{2}([\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}})''_{|\tau_1=t_2}(t_1-\tau_1)\right\}^{3/2}\right|,$$

$$t_2=\tau_1+\theta_2(t_1-\tau_1),\ 0<\theta_2<1.$$

Because $\alpha(t_1)=[t_1(1+\alpha_0(t_1))]^{\omega}$, we get:

$$|P_2(t,\tau)-P(t,\tau)|\le\frac{1}{2\sqrt{\pi}}\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{(1-2\omega)^{3/2}[t_1(1+\alpha_0(t_1))]^{\frac{3}{2}-3\omega}[\tau_1(1+\alpha_0(\tau_1))]^{\frac{-1-2\omega}{2}}}{\delta^{3/2}(\omega)[t_1(1+\alpha_0(t_1))]^{-3\omega}\left(1+\alpha_0(t_1)+t_1\alpha_0'(t_1)\right)^{3/2}(t_1-\tau_1)^{3/2}}\times$$

$$\times\left|\left(1+\alpha_0(\tau_1)+\tau_1\alpha_0'(\tau_1)\right)-\left(1+\alpha_0(t_1)+t_1\alpha_0'(t_1)\right)^{3/2}\left\{1+\omega[t_1(1+\alpha_0(t_1))]^{-1}\left(1+\alpha_0(t_1)+t_1\alpha_0'(t_1)\right)(t_1-\tau_1)-\right.\right.$$

$$\left.\left.-\frac{1}{2}\left(1+\alpha_0(t_1)+t_1\alpha_0'(t_1)\right)^{-1}\left(2\alpha_0'(t_2)+t_2\alpha_0''(t_2)\right)(t_1-\tau_1)\right\}^{3/2}\right|.$$

So, we finally get: $|P'(t_1,\tau_1)-P_2'(t_1,\tau_1)|$

$$\le\frac{1}{2\sqrt{\pi}}\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{(1-2\omega)^{3/2}}{\delta^{3/2}(\omega)}\cdot\frac{2^{\omega}t_1^{1-\omega}\alpha_0(t_1)}{(t_1-\tau_1)^{3/2}}\le\overline{M}\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{t_1^{1-\omega+\beta}}{(t_1-\tau_1)^{3/2}}.$$

Lemma 1 is proved.

Lemma 2. if the conditions of lemma 1 are satisfied, then we have

$$|Q'(t_1,\tau_1)-Q_2'(t_1,\tau_1)|\le M_1\frac{t_1^{2\omega+\beta}}{t_1-\tau_1}+M_2t_1^{2\omega-1}.$$

Proof of Lemma 2.
We have:

$$|Q'(t_1,\tau_1)-Q_2'(t_1,\tau_1)|=\left|\frac{t_1\tau_1[\alpha(\tau_1)]^{-2}}{4(t_1-\tau_1)}-(1-2\omega)\frac{[\alpha(t_1)]^{\frac{1-2\omega}{\omega}}[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}}{4\left([\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}\right)}\right|=$$

$$=\frac{1-2\omega}{4}\frac{[\alpha(t_1)]^{\frac{1-2\omega}{\omega}}[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}}{[\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}}\left|\frac{t_1\tau_1[\alpha(\tau_1)]^{-2}\left([\alpha(t_1)]^{\frac{1-2\omega}{\omega}}-[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}\right)}{(1-2\omega)(t_1-\tau_1)[\alpha(t_1)]^{\frac{1-2\omega}{\omega}}[\alpha(\tau_1)]^{\frac{1-2\omega}{\omega}}}-1\right|\le$$

$$\leq \frac{1-2\omega}{4}\frac{\left[t_1(1+\alpha_0(t_1))\right]^{1-2\omega}\left[\tau_1(1+\alpha_0(\tau_1))\right]^{1-2\omega}}{\left[t_1(1+\alpha_0(t_1))\right]^{1-2\omega}-\left[\tau_1(1+\alpha_0(\tau_1))\right]^{1-2\omega}}\times$$

$$\times\left| t_1\tau_1\left[t_1(1+\alpha_0(t_1))\right]^{2\omega-1}\left[\tau_1(1+\alpha_0(\tau_1))\right]^{-1}\frac{\left\{\left(\left[\alpha(\tau_1)\right]^{\frac{1-2\omega}{\omega}}\right)'_{|\tau_1=t_1}-\frac{1}{2}\left(\left[\alpha(\tau_1)\right]^{\frac{1-2\omega}{\omega}}\right)''_{|\tau_1=t_2}(t_1-\tau_1)\right\}}{(1-2\omega)}-1\right|\leq$$

$$\leq \frac{(1-2\omega)2^{2-2\omega}}{4\delta(\omega)}\frac{t_1^{2\omega}}{(t_1-\tau_1)}(\alpha_0(t_1)+t_1\alpha_0'(t_1))+\frac{(1-2\omega)\omega}{4\delta(\omega)}2^{2\omega-1}t_1^{2\omega-1}-\frac{(1-2\omega)}{4\delta(\omega)}2^{2\omega}t_1^{2\omega}\leq$$

$$\leq M_1\frac{t_1^{2\omega+\beta}}{t_1-\tau_1}+M_2t_1^{2\omega-1}.$$

Lemma 2 is proved.

Proof of Theorem 2. First, we establish the following inequality:

$$P_2'(t_1,\tau_1)=\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\cdot\frac{\tau_1^{-1/2}t_1^{3/2}\left[\tau_1(1+\alpha_0(\tau_1))\right]^{-\omega}}{(t_1-\tau_1)^{3/2}}\leq M_3(\omega)\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\cdot\frac{t_1^{1-\omega}}{(t_1-\tau_1)^{3/2}}.$$

For those parameter values ω and $0<\tau<t<\infty,$ at which $Q'(t_1,\tau_1)\geq Q_2'(t_1,\tau_1)$ the required estimate follows from these inequalities:

$$\left|K'(t_1,\tau_1)-K_2'(t_1,\tau_1)\right|\leq\left|\left(P'(t_1,\tau_1)-P_2'(t_1,\tau_1)\right)\exp\left\{-Q'(t_1,\tau_1)\right\}\right|+$$

$$+\left|P_2'(t_1,\tau_1)\exp\left\{-Q'(t_1,\tau_1)\right\}\left(1-\exp\left\{-Q_2'(t_1,\tau_1)+Q'(t_1,\tau_1)\right\}\right)\right|\leq$$

$$\leq\left|P'(t_1,\tau_1)-P_2'(t_1,\tau_1)\right|\exp\left\{-Q'(t_1,\tau_1)\right\}+\left|P_2'(t_1,\tau_1)\left(Q_2'(t_1,\tau_1)-Q'(t_1,\tau_1)\right|\exp\left\{-Q'(t_1,\tau_1)\right\}.\right.$$

Thus, taking into account Lemmas 1 - 2, we have:

$$\left|K'-K_2'\right|\leq\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\left\{\overline{M}\frac{t_1^{1-\omega+\beta}}{(t_1-\tau_1)^{3/2}}+M_3\frac{t_1^{1-\omega+\beta}}{(t_1-\tau_1)^{3/2}}\left(M_1\frac{t_1^{2\omega+\beta}}{t_1-\tau_1}+M_2t_1^{2\omega-1}\right)\right\}exp\left(-Q\right)\leq$$

$$\leq\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{t_1^{3/2-2\omega}}{\tau_1^{3/2-\omega}(t_1-\tau_1)^{1/2}}\left(\overline{M}\frac{t_1^{\omega+\beta-1/2}}{t_1-\tau_1}+\overline{M}_1\frac{\tau_1^{3/2-\omega}t_1^{3\omega+\beta-1/2}}{(t_1-\tau_1)^2}+\overline{M}_2\frac{\tau_1^{3/2-\omega}t_1^{3\omega-3/2}}{t_1-\tau_1}\right)\exp(-Q)\leq$$

$$\leq\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}}\frac{t_1^{3/2-2\omega}}{\tau_1^{3/2-\omega}(t_1-\tau_1)^{1/2}}\left(\frac{t_1}{t_1-\tau_1}exp\left(-Q/2\right)\cdot\overline{M}t_1^{\omega+\beta-3/2}\tau_1^{3/2-\omega}+\right.$$

$$\left.+\frac{t_1^2}{(t_1-\tau_1)^2}exp\left(-Q/2\right)\cdot\overline{M}_1\tau_1^{3/2-\omega}t_1^{3\omega+\beta-5/2}+\frac{t_1}{t_1-\tau_1}exp\left(-Q/2\right)\cdot\overline{M}_2\tau_1^{3/2-\omega}t_1^{3\omega-5/2}\right)\exp\left(-Q/2\right)\leq$$

$$\le C(\omega)\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}} \frac{t_1^{-\omega}}{\sqrt{t_1-\tau_1}}\exp\{-Q/2\}.$$

The validity of inequality (21) means that the kernel $K_2'(t_1,\tau_1)-K'(t_1,\tau_1)$ has a weak singularity and the following limiting relation holds:

$$\lim_{t_1\to 0}\int_0^{t_1}\left[\frac{\alpha(\tau_1)}{\alpha(t_1)}\right]^{\frac{3/2-\omega}{\omega}} \cdot \frac{t_1^{3/2-2\omega}}{\tau_1^{3/2-\omega}(t_1-\tau_1)^{1/2}}\left[\exp\{-Q'(t_1,\tau_1)/2\}+\exp\{-Q_2'(t_1,\tau_1)/2\}\right]d\tau_1 \le$$

$$\le \lim_{t_1\to 0}\int_0^{t_1}\frac{t_1^{-\omega}}{(t_1-\tau_1)^{1/2}}d\tau_1 = 0\,.$$

This means that equation (18) is indeed characteristic for equation (15). Thus, Theorem 2 and thus, Theorem 1 is proved.

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