МАТЕМАТИКА

MATHEMATICS

МРНТИ 42B20, 42B25
УДК 512.51

https://doi.org/10.51889/2022-2.1728-7901.01

CALDERON-ZIGMUND SINGULAR INTEGRAL IN THE MORREY-TYPE SPACES WITH VARIABLE EXPONENTS

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Abstract

In this paper we consider global Morrey spaces $GM_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponents p(.), θ(.), where $\Omega \subset R^n$ is an unbounded set. The questions of boundedness of the singular integral operator and its commutator in these spaces are investigated. We give the conditions for the variable exponent $\left(\theta_1(.),\theta_2(.)\right)$ and for the functions (w_1(.),w_2 (.)) under which the singular integral operator T will be bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$. The same conditions for the boundedness of the commutator of the singular integral operator in these spaces are obtained.. In the case when the exponents p,θ are constant numbers, the questions of boundedness of the singular integral operator and its commutator in global spaces were previously studied by other authors. There are also well-known results on the boundedness of a singular integral operator in the global Morrey-type spaces with variable exponents when the set $\Omega \subset R^n$ is bounded.

Keywords: singular integral operator, commutator, boundedness, Morrey-type spaces, variable exponents, unbounded set, Holder inequality.

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КӨРСЕТКІШТЕРІ АЙНЫМАЛЫ МОРРИ КЕҢІСТІКТЕРІНДЕГІ КАЛЬДЕРОН-ЗИГМУНД СИНГУЛЯРЛЫҚ ИНТЕГРАЛЫ

Біз бұл жұмыста көрсеткіштері p(.), θ(.) айнымалы глобальді Морри типтес кеңістіктер $GM_{p(.),\theta(.),w(.)}(\Omega)$ қарастырамыз, мұндағы $\Omega \subset R^n$ шенелмеген жиын. Осы кеңістіктерде Кальдерона-Зигмунд операторы және оның коммутаторының шенелгендігі туралы мәселелер қарастырылады. $\left(\theta_1(.),\theta_2(.)\right)$ көрсеткіштері мен (w_1(.),w_2 (.)) функцияларына сингулярлық интегральдық оператор T $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ кеңістігінен $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$ кеңістігіне шенелгендігін қамтамасыз ететін шарттар алынды. Дәл осы сияқты шарттар көрсетілген кеңістіктерде сигулярлық интегралдық оператордың коммутаторына да алынды. p,θ көрсеткіштері тұрақты болғанда, сингулярлық интеграл және оның коммутаторының глобальді Морри типтес кеңістіктердегі шенелгендігін басқа авторлар зерттеген. $\Omega \subset R^n$ жиыны шенелген болған жағдайда да глобальді Морри типтес кеңістіктердегі сингулярлық интегралдық операторлдың шенелгендік шарттары да белгілі.

Ключевые сөздер: сингулярлық интегральдық оператор, коммутатор, шенелгендік, Морри типтес кеңістіктер, айнымалы көрсеткіш, шенелмеген жиын, Гельдер теңсіздігі.

Аннотация

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СИНГУЛЯРНЫЙ ИНТЕГРАЛ КАЛЬДЕРОНА-ЗИГМУНДА В ПРОСТРАНСТВАХ ТИПА МОРРИ С ПЕРЕМЕННЫМИ ПОКАЗАТЕЛЯМИ

В данной работе мы рассматриваем глобальные пространства Морри $GM_{p(.),\theta(.),w(.)}(\Omega)$ с переменными показателями p(.), θ(.), где $\Omega \subset R^n$ является неограниченным множеством. Исследуются вопросы ограниченности сингулярного интегрального оператора Кальдерона-Зигмунда и его коммутатора в этих пространствах. Приведены условия для переменного показателя степени $(\theta_1(.), \theta_2(.))$ и функций (w_1(.),w_2 (.)) при которых сингулярный интегральный оператор T будет ограничен из $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ в $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$. Получены такие же условия ограниченности коммутатора сингулярного интегрального оператора в этих пространствах. В случае, когда показатели p, θ являются постоянными числами, вопросы ограниченности сингулярного интегрального оператора и его коммутатора в глобальных пространствах ранее изучались другими авторами. Известны также результаты об ограниченности сингулярного интегрального оператора в глобальных пространствах типа Морри с переменными показателями, когда множество $\Omega \subset R^n$ ограничено.

Ключевые слова: сингулярный интегральный оператор, коммутатор, ограниченность, пространства типа Морри, переменный показатель, неограниченное множество, неравенство Гельдера.

1. Introduction

The Morrey space $M_{p,\lambda}$ first appeared in [1] in connection with the study of solutions of partial differential equations. The boundedness of the classical integral operators of harmonic analysis in the global Morrey-type spaces $GM_{p(.),\theta(.),w(.)}(\Omega)$ with constant exponents p, θ are well studied (see, for example [2]).

Lebesgue space with variable exponent and the boundedness of classical operators in these spaces are studied in ([3]-[4]).

The Morrey-type space $L_{p(.),\lambda(.)}$ with variable exponents is also well studied (see [5]). The generalized Morrey-type space $M_{p(.),w(.)}(\Omega)$ with variable exponent in the case of a bounded set $\Omega \subset R^n$ was introduced and studied in [6] and in the case of an unbounded set Ω was studied in [7].

The Calderón-Zygmund singular integral operator is defined by

$$Tf(x) = \int_{R^n} K(x,y) f(y) dy.$$

Here the kernel is a K(x,y)-continuous function on Ω×Ω and satisfies the following conditions

$$|K(x,y)| < C|x-y|^n \quad for\ all \quad x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C \frac{|y-z|^\delta}{|x-y|^{n+\delta}}, \quad \delta > 0, \qquad if \quad |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C \frac{|x-\xi|^\delta}{|x-y|^{n+\delta}}, \quad \delta > 0, \qquad if \quad |x-y| > 2|x-\xi|.$$

Boundedness of the Calderon-Zygmund singular integral operator in generalized Morrey-type spaces with variable exponent was studied in [6] in the case of a bounded set

$\Omega \subset R^n$, in the case of an unbounded set $\Omega \subset R^n$ in [7].

Here and below, we denote by $B(x,r)$ the ball centered at the point $x \in R^n$ and radius $r > 0$, $\widetilde{B(x,r)} = B(x,r) \cap \Omega$, $\Omega \subset R^n$.

The space $BMO(\Omega)$ is defined as the space of all integrable functions f with finite norm

$$\|f\|_{BMO} = \|f\|_* = sup_{x \in \Omega, r > 0} |B(x,r)|^{-1} \int_{\widetilde{B(x,r)}} \left| f(y) - f_{\widetilde{B(x,r)}} \right| dy,$$

where $f_{\widetilde{B(x,r)}} = \left|\widetilde{B(x,r)}\right|^{-1} \int_{\widetilde{B(x,r)}} f(y) dy.$

Let $b \in BMO(\Omega)$. The commutator of is defined by Calderón-Zygmund singular integral operator is defined by

$$[b,T]f = T(bf) - b(Tf) = \int_{R^n} K(x,y)(b(y)-b(x))f(y)dy.$$

Boundedness of the commutator's Calderón-Zygmund singular integral operator in the generalized Morrey type spaces was investigated in [8].

Boundedness of the Hardy operator in the variavle exponent Lebesgue spaces was studied in [9].

2. Basic definitions. Preliminary results

Let $p(x)$ be a measurable function on $\Omega \subset R^n$ with values $(1,\infty)$. Let

$$1 < p_- \le p(x) \le p_+ < \infty, \qquad (2.1)$$

where

$$p_- = p_-(\Omega) = essinf_{x\in\Omega} p(x),\ p_+ = p_+(\Omega) = essup_{x\in\Omega} p(x).$$

We denote by $L_{p(.)}(\Omega)$ the space of all measurable functions on Ω, such that

$$J_{p(.)}(f) = \int_\Omega |f(x)|^{p(x)} dx < \infty,$$

where the norm is defined as follows

$$\|f\|_{p(.)} = \inf\left\{\eta > 0,\ J_{p(.)}\left(\frac{f}{\eta}\right) \le 1\right\}.$$

This is a Banach space (see [8]). The conjugate exponent p' is defined by the formula

$$p'(x) = \frac{p(x)}{p(x)-1}.$$

The Holder inequality for variable exponents $p(.)$, $p'(.)$ (see, for example [8]):

$$\int_\Omega |f(x)g(x)|dx \le C(p)\|f\|_{L_p(\Omega)} \|g\|_{L_{p'}(\Omega)}.$$

Let $P(\Omega)$ be the set of measurable functions $p(x)$ for which $p\colon \Omega \to [1,\infty)$, $P^{log}(\Omega)$ is the set of all measurable functions $p(x)$ satisfying the local logarithmic condition:

$$|p(x)-p(y)| \le \frac{A_p}{-ln|x-y|}\ ,\ |x-y| \le \frac{1}{2},\ x,y \in \Omega\,,$$

where the constant number A_p does not depend on x and y. $\boldsymbol{P}^{log}(\Omega)$ is the set of all measurable functions $p(x)$ satisfying (2.1) and the local logarithmic condition. In the case where Ω is an unbounded set, we denote by $\boldsymbol{P}^{log}_\infty(\Omega)$ a subset of the set $\boldsymbol{P}^{log}(\Omega)$ satisfying logarithmic condition at infinity:

$$|p(x)-p(\infty)| \le \frac{A_\infty}{\ln(2+|x|)},\quad x \in R^n.$$

Let Ω be a bounded open set, $p \in \boldsymbol{P}^{log}(\Omega)$ and $\lambda(x)$ be a measurable function on Ω with values on $[0,n]$. The Morrey spaces $L_{p(.),\lambda(.)}$ with variable exponents $p(.), \lambda(.)$ were introduced in [5] with the norm

$$\|f\|_{L_{p(.),\lambda(.)}} = sup_{x\in\Omega, t>0}\, t^{-\frac{\lambda(x)}{p(x)}} \|f\|_{B(x,t)}.$$

Let $w(x,r)$ be a positive measurable function on $\Omega\times(0,l)$, where Ω is a bounded set, $l=diam\Omega$. The generalized Morrey spaces $M_{p(.),w(.)}(\Omega)$ with variable exponent on a bounded domain $\Omega\subset R^n$ were defined in [6] with norm

$$\|f\|_{M_{p(.),w(.)}}=sup_{x\in\Omega,r>0}\frac{r^{-\frac{n}{p(x)}}}{w(x,r)}\|f\|_{L_{p(.)}(B(x,r))}.$$

Let $w(x,r)$ be a positive measurable function on $\Omega\times(0,l)$, where Ω is a bounded set, $l=diam\Omega$, a measurable function $\theta(r):(0,l)\to[1,\infty]$. The Morrey spaces $M_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponents t on a bounded domain $\Omega\subset R^n$ were defined in [8] with the norm

$$\|f\|_{M_{p(.),\theta(.),w(.)}}=sup_{x\in\Omega,}\left\|w(x,r)r^{-\frac{n}{p(x)}}\|f\|_{L_{p(.)}(B(x,r))}\right\|_{L_{\theta(.)}(0,l)}.$$

Let $w(x,r)$ be a positive measurable function on $\Omega\times(0,\infty)$, where Ω is a unbounded set. The generalized Morrey spaces $M_{p(.),w(.)}(\Omega)$ with variable exponent on an unbounded domain $\Omega\subset R^n$ were defined in [7] with norm

$$\|f\|_{M_{p(.),w(.)}}=sup_{x\in\Omega,r>0}\frac{\|f\|_{L_{p(.)}(B(x,r))}}{w(x,r)}.$$

We introduce global Morrey-type spaces with variable exponents on unbounded domains. Let's put

$$\eta_p(x,r)=\begin{cases}\frac{n}{p(x)}, & if\ r\le 1\\ \frac{n}{p(\infty)}, & if\ r>1.\end{cases}$$

Definition 1. Let $p\in\boldsymbol{P}_{\infty}^{log}(\Omega)$, $w(x,r)$ be a positive measurable function on $\Omega\times(0,\infty)$, where $\Omega\subset R^n$ is a unbounded set, a measurable function $\theta(r):(0,\infty)\to[1,\infty]$. The global Morrey-type spaces $GM_{p(.),\theta(.),w(.)}(\Omega)$ with variable exponents $p(.),\theta(.)$ defined as the set of functions with finite norm

$$\|f\|_{GM_{p(.),\theta(.),w(.)}(\Omega)}=sup_{x\in\Omega,}\left\|w(x,r)r^{-\eta_p(x,r)}\|f\|_{L_{p(.)}(B(x,r))}\right\|_{L_{\theta(.)}(0,\infty)},$$

for $1\le\theta(r)<\infty$, with finite norm

$$\|f\|_{GM_{p(.),\infty,w(.)}(\Omega)}=sup_{x\in\Omega,r>0}w(x,r)r^{-\eta_p(x,r)}\|f\|_{L_{p(.)}(B(x,r))},$$

for $\theta(r)=\infty$.

Note that, the space $GM_{p(.),\infty,w(.)}(\Omega)$ coincides with the generalized Morrey-type space $M_{p(.),w_1(.)}(\Omega)$ with variable exponent, where $w_1(x,r)=\frac{r^{\eta_p(x,r)}}{w(x,r)}$.

In the case of $w(x,r)=r^{\eta_p(x,r)-\frac{\lambda(x)}{p(x)}}$ we denote the indicated space by via $GM_{p(.),\theta(.)}^{\lambda(.)}(\Omega)$:

$$GM_{p(.),\theta(.)}^{\lambda(.)}(\Omega)=GM_{p(.),\theta(.),w(.)}|_{w(x,r)=r^{\eta_p(x,r)-\frac{\lambda(x)}{p(x)}}}(\Omega),$$

$$\|f\|_{GM_{p(.),\theta(.)}^{\lambda(.)}(\Omega)}=sup_{x\in\Omega,}\left\|r^{-\frac{\lambda(x)}{p(x)}}\|f\|_{L_{p(.)}(B(x,r))}\right\|_{L_{\theta(.)}(0,\infty)}.$$

If $p(x) = p = const$, $\theta(r) = \theta = const$, then the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ coincides with the well-known ordinary global Morrey space $GM_{p,\theta,w}(\Omega)$.

The following condition gives a sufficient condition under which the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ is not trivial:

$$sup_{x\in\Omega,}\|w(x,r)\|_{L_{\theta(.)}(0,\infty)} < \infty.$$

In this case the space $GM_{p(.),\theta(.),w(.)}(\Omega)$ contains bounded functions.

We need the next theorem, which was proved in [7].

Theorem 1. Let $p \in \boldsymbol{P}_{\infty}^{log}(\Omega)$. Then

$$\|Tf\|_{L_{p(.)}(B(x,t))} \leq Ct^{\eta_p(x,t)} \int_t^{\infty} r^{-\eta_p(x,r)-1} \|f\|_{L_{p(.)}(B(x,r))}dr,$$

where C is independent of $f, x \in \Omega$ and $t > 0$.

The next theorem was proved in [10].

Theorem 2. Let $p \in \boldsymbol{P}_{\infty}^{log}(\Omega)$ and $b \in BMO(\Omega)$. Then

$$\|[b,T]\|_{L_{p(.)}(B(x,t))} \leq Cb_* t^{\eta_p(x,t)} \int_t^{\infty} (1 + ln\frac{r}{t}) r^{-\eta_p(x,r)-1} \|f\|_{L_{p(.)}(B(x,r))}dr,$$

where C is independent of $f, x \in \Omega$ and $t > 0$.

3. Main Results

Theorem 3. Let $p \in \boldsymbol{P}_{\infty}^{log}(\Omega)$ and $\theta_1(r)$, $\theta_2(r)$ are measurable functions on R_+, such that $1 < \theta_{1-} \leq \theta(r) \leq \theta_{1+} < \infty$, $1 < \theta_{2-} \leq \theta(r) \leq \theta_{2+} < \infty$. Suppose that the measurable functions $w_1(x,r)$ and $w_2(x,r)$ satisfy the condition

$$A = sup_{x\in\Omega} \left\| w_2(x,r) \left\| \frac{1}{w_1(x,t)t} \right\|_{L_{\theta'_1(.)}(r,\infty)} \right\|_{L_{\theta_2}(.)(0,\infty)} < \infty.$$

Then the Calderon-Zygmund integral operator T is bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$.

Proof. Using the Theorem 1 and the Holder inequality with exponents $\theta_1(.)$, $\theta'_1(.)$, we have:

$$\|Tf\|_{GM_{p(.),\theta_2(.),w_2(.)}(\Omega)} =$$

$$sup_{x\in\Omega,} \left\| w_2(x,r) r^{-\eta_p(x,r)} \|Tf\|_{L_{p(.)}(B(x,r))} \right\|_{L_{\theta_2(.)}(0,\infty)} \leq$$

$$\leq Csup_{x\in\Omega,} \left\| w_2(x,r) \int_r^{\infty} t^{-\eta_p(x,t)-1} \|f\|_{L_{p(.)}(B(x,t))}dt \right\|_{L_{\theta_2(.)}(0,\infty)} =$$

$$= Csup_{x\in\Omega,} \left\| w_2(x,r) \int_r^{\infty} w_1(x,t) t^{-\eta_p(x,t)} \|f\|_{L_{p(.)}(B(x,t))} \frac{1}{w_1(x,t)t} dt \right\|_{L_{\theta_2(.)}(0,\infty)}$$

$$\leq Csup_{x\in\Omega} \left\| w_2(x,r) \left\| \frac{1}{w_1(x,t)t} \right\|_{L_{\theta'_1(.)}(r,\infty)} \left\| w_1(x,t) t^{-\eta_p(x,t)} \|f\|_{L_{p(.)}(B(x,t))} \right\|_{L_{\theta_1(.)}(r,\infty)} \right\|_{L_{\theta_2(.)}(0,\infty)} \leq$$

$$\leq Csup_{x\in\Omega} \left\| w_2(x,r) \left\| \frac{1}{w_1(x,t)t} \right\|_{L_{\theta'_1(.)}(r,\infty)} \right\|_{L_{\theta_{2(.)}}(0,\infty)} sup_{x\in\Omega,} \left\| w_1(x,t) t^{-\eta_p(x,t)} \|f\|_{L_{p(.)}(B(x,t))} \right\|_{L_{\theta_1(.)}(0,\infty)}$$

$$=$$

$$= CA\|f\|_{GM_{p(.),\theta_1(.),w_1(.)}(\Omega)}.$$

This means that, the Calderon-Zygmund integral operator T is bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$. The Theorem 3 is proved.

Theorem 4. Let $p \in \boldsymbol{P}_{\infty}^{log}(\Omega)$, $b \in BMO(\Omega)$ and $\theta_1(r)$, $\theta_2(r)$ are measurable functions on R_+, such that $1 < \theta_{1\_} \le \theta(r) \le \theta_{1+} < \infty$, $1 < \theta_{2\_} \le \theta(r) \le \theta_{2+} < \infty$. Suppose that the measurable functions $w_1(x,r)$ and $w_2(x,r)$ satisfy the condition

$$B = sup_{x\in\Omega}\left\|\frac{w_2(x,r)}{r}\left\|\frac{1}{w_1(x,t)}\right\|_{L_{\theta'_1(.)}(r,\infty)}\right\|_{L_{\theta_2}(.)(0,\infty)} < \infty.$$

Then the commutator $[b,T]$ is bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$.

Proof. Using the Theorem 2, the inequality $1 + ln\frac{t}{r} < \frac{t}{r}$ for $0 < r < t$ and the Holder inequality with exponents $\theta_1(.)$, $\theta'_1(.)$, we have:

$$\|[b,T]f\|_{GM_{p(.),\theta_2(.),w_2(.)}(\Omega)} =$$

$$= sup_{x\in\Omega,}\left\|w_2(x,r)r^{-\eta_p(x,r)}\|[b,T]f\|_{L_{p(.)}(B(x,r))}\right\|_{L_{\theta_2(.)}(0,\infty)} \le$$

$$\le Csup_{x\in\Omega,}\left\|w_2(x,r)\int_r^{\infty}(1+ln\frac{t}{r})t^{-\eta_p(x,t)-1}\|f\|_{L_{p(.)}(B(x,t))}dt\right\|_{L_{\theta_2(.)}(0,\infty)} \le$$

$$\le Csup_{x\in\Omega,}\left\|\frac{w_2(x,r)}{r}\int_r^{\infty}t^{-\eta_p(x,t)}\|f\|_{L_{p(.)}(B(x,t))}dt\right\|_{L_{\theta_2(.)}(0,\infty)} =$$

$$= Csup_{x\in\Omega,}\left\|\frac{w_2(x,r)}{r}\int_r^{\infty}w_1(x,t)t^{-\eta_p(x,t)}\|f\|_{L_{p(.)}(B(x,t))}\frac{1}{w_1(x,t)}dt\right\|_{L_{\theta_2(.)}(0,\infty)}$$

$$\le Csup_{x\in\Omega}\left\|\frac{w_2(x,r)}{r}\left\|\frac{1}{w_1(x,t)}\right\|_{L_{\theta'_1(.)}(r,\infty)}\left\|w_1(x,t)t^{-\eta_p(x,t)}\|f\|_{L_{p(.)}(B(x,t))}\right\|_{L_{\theta_1(.)}(r,\infty)}\right\|_{L_{\theta_2(.)}(0,\infty)}$$

$$\le Csup_{x\in\Omega}\left\|\frac{w_2(x,r)}{r}\left\|\frac{1}{w_1(x,t)}\right\|_{L_{\theta'_1(.)}(r,\infty)}\right\|_{L_{\theta_2}(.)(0,\infty)} sup_{x\in\Omega,}\left\|w_1(x,r)r^{-\eta_p(x,r)}\|f\|_{L_{p(.)}(B(x,r))}\right\|_{L_{\theta_1(.)}(0,\infty)}$$

$$= CB\|f\|_{GM_{p(.),\theta_1(.),w_1(.)}(\Omega)}.$$

This means that, the commutator $[b,T]$ is bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$. The Theorem 4 is proved.

4. Conclusion

We have obtained the sufficient conditions of the boundedness of the Calderon-Zygmund singular integral operator and its commutator in the global Morrey-type spaces with variable exponents.

We gave the conditions for the variable exponent $(\theta_1(.), \theta_2(.))$ and for the functions $(w_1(.), w_2(.))$ under which the singular integral operator T will be bounded from $GM_{p(.),\theta_1(.),w_1(.)}(\Omega)$ to $GM_{p(.),\theta_2(.),w_2(.)}(\Omega)$.

References:

1 Morrey C.B., On the solutions of quasi-linear elliptic partial differential equations. // Trans.Am.Math.Soc.43 (1938), 126-166.

2 Burenkov. V, Guliyev. V, Serbetci. A, Tararykova. T. Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces. // Eurasian Mathematical Journal, Volume 1, Number 1(2010), 32-53.

3 Stein E. Harmonic analysis: real variable methods, orthogonality, and oscillatory integrals. // Princeton Univ.Press, Princeton, NJ, 1993.

4 Almeida A, Samko. S, Fractional and hypersingular operators in variable exponent spaces on metric measure spaces. //Meditter.J.Math.,6(2009), 215-232.

5 Almeida A, Hasanov. J., Samko. S, Maximal and potential operators in variable exponent Morrey spaces . // Georgian Mathematical Journal, Vol.15 (2008), no. 2, 195-208.

6 Guliyev, V, Samko S, Hasanov J. Boundedness of the maximal, potential type and singular integral operators in the generalized variable exponent Morrey spaces. // Math.Scand. – 2010. – Vol.107, – Pp. 285-304. https://doi.org/10.7146/math.scand.a-15156

7 Guliyev. V, Samko. S, Maximal, potential, and singular operators in the generalized variable exponent Morrey spaces on unbounded sets. //J.Math.Sciences, Vol.193(2013),228-247. https://doi.org/10.1007/s10958-013-1449-8

8 Guliyev. V, Hasanov. J, Badalov A., Maximal and singular integral operators and their commutators on generalized weighted Morrey spaces with variable exponent.//Math.Ineq.Appl, Vol.21(2018), 41-61. DOI:10.7153/mia-2018-21-04

9 Edmunds. D , Kokilasvili. V, Meskhi. A On the boundedness and compactness of weighted Hardy operators in spaces $L_{p(x)}$. // Georg.Math.J.12(2005), Num.1, 27-44.

10 L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka Lebesgue and Sobolev spaces with variable exponents // Monograpgh. 2010. Pp. 1-493. http://citeseerx.ist.psu.edu/viewdoc/ download?doi=10.1.1.581. 2463&rep= rep1&type =pdf