

Y. Wang^{1*}, D. Sydyk¹, S. Omirbekov¹

¹Nazarbayev University, Astana, Kazakhstan

*e-mail: yanwei.wang@nu.edu.kz

A LEGENDRE-FENCHEL TRANSFORM FRAMEWORK FOR MATHEMATICAL MODELING OF NONLINEAR CAPILLARY FLOW OF GENERALIZED NEWTONIAN FLUIDS

Abstract

A Legendre-Fenchel transform framework is developed for pressure-driven capillary flow of monotone generalized Newtonian fluids. A monotone nonlinear flow curve defines a convex dissipation potential, and its Legendre-Fenchel transform gives a complementary potential in the stress variable. Since the stress distribution is known explicitly in fully developed capillary flow, the velocity profile can be obtained from this complementary potential without requiring the inverse constitutive relation to be written explicitly. This reformulates nonlinear capillary-flow modeling from a model-by-model constitutive inversion problem into a convex-duality construction. The method recovers the shear-rate-parametrized Carreau-Yasuda solution, gives compact average-velocity relations for circular and planar geometries, and extends to several generalized Newtonian models, including Papanastasiou-regularized yield-stress fluids. In the ideal yield-stress limit, plug formation follows naturally from the subdifferential structure of the dissipation potential and the corresponding flat interval of the complementary potential. The framework provides a unified mathematical structure for analytical and semi-analytical pressure-drop-flow-rate relations in nonlinear capillary flow.

Keywords: Legendre-Fenchel transform; capillary flow; generalized Newtonian fluids; convex duality; non-Newtonian rheology; pressure-driven flow; yield-stress fluids.

Я. Ванг¹, Д. Сыдык¹, С. Омирбеков¹

¹Назарбаев Университет, г.Астана, Казахстан

ПРЕОБРАЗОВАНИЕ ЛЕЖАНДРА-ФЕНХЕЛЯ ДЛЯ МАТЕМАТИЧЕСКОГО МОДЕЛИРОВАНИЯ НЕЛИНЕЙНОГО КАПИЛЛЯРНОГО ТЕЧЕНИЯ ОБОБЩЁННЫХ НЬЮТОНОВСКИХ ЖИДКОСТЕЙ

Аннотация

В работе разработана схема преобразования Лежандра-Фенхеля для напорного капиллярного течения монотонных обобщённых ньютоновских жидкостей. Монотонная нелинейная кривая течения задаёт выпуклый потенциал диссипации, а её преобразование Лежандра-Фенхеля формирует дополнительный потенциал в переменной напряжения. Поскольку распределение напряжения в полностью развитом капиллярном течении известно из уравнения импульса, профиль скорости может быть получен непосредственно из дополнительного потенциала без явного обращения определяющего соотношения. Такой подход переводит моделирование нелинейного капиллярного течения из задачи обращения отдельных реологических моделей в единую конструкцию выпуклой двойственности. Метод воспроизводит параметризованное по скорости сдвига решение Карро-Ясуды, даёт компактные соотношения для средней скорости в круглой и плоской геометриях и распространяется на ряд обобщённых ньютоновских моделей, включая регуляризованные модели сред с пределом текучести. В пределе идеальной среды с пределом текучести образование пробковой зоны естественно следует из субдифференциальной структуры потенциала диссипации. Предложенная схема обеспечивает единое математическое описание аналитических и полуаналитических зависимостей между перепадом давления и расходом в нелинейном капиллярном течении.

Ключевые слова: преобразование Лежандра-Фенхеля; капиллярное течение; обобщённые ньютоновские жидкости; выпуклая двойственность; неньютоновская реология; напорное течение; жидкости с пределом текучести.

Я. Ванг¹, Д. Сыдық¹, С. Өмірбеков¹

¹Назарбаев университеті, Астана қ., Қазақстан

**ЖАЛПЫЛАНҒАН НЬЮТОНДЫҚ СҰЙЫҚТАРДЫҢ СЫЗЫҚТЫҚ ЕМЕС
КАПИЛЛЯРЛЫҚ АҒЫНЫН МАТЕМАТИКАЛЫҚ МОДЕЛЬДЕУГЕ АРНАЛҒАН
ЛЕЖАНДР-ФЕНХЕЛЬ ТҮРЛЕНДІРУІ**

Аңдатпа

Бұл жұмыста монотонды жалпыланған ньютондық сұйықтардың қысым әсерінен болатын капиллярлық ағынын сипаттау үшін Лежандр–Фенхель түрлендіруіне негізделген тәсіл ұсынылады. Монотонды сызықтық емес ағын қисығы дөңес диссипация потенциалын анықтайды, ал оның Лежандр–Фенхель түрлендіруі кернеу айнымалысындағы қосымша потенциалды береді. Толық дамыған капиллярлық ағында кернеудің таралуы импульс теңдеуінен белгілі болғандықтан, жылдамдық профилін анықтаушы теңдеуді айқын түрде кері шығармай-ақ осы қосымша потенциал арқылы алуға болады. Осылайша, сызықтық емес капиллярлық ағынды модельдеу жеке реологиялық модельдерді кері айналдыру мәселесінен дөңес дуалдылыққа негізделген бірыңғай математикалық құрылымға көшіріледі. Әдіс Карро-Ясуда моделінің ығысу жылдамдығымен параметрленген шешімін қалпына келтіреді, дөңгелек және жазық геометриялар үшін орташа жылдамдықтың ықшам қатынастарын береді және Папанастасиу бойынша регуляризацияланған аққыштық шегі бар сұйықтарды қоса алғанда бірнеше жалпыланған ньютондық модельдерге қолданылады. Идеал аққыштық шегі бар сұйық шегінде тығын тәрізді аймақтың пайда болуы диссипация потенциалының субдифференциалдық құрылымынан табиғи түрде шығады. Ұсынылған тәсіл сызықтық емес капиллярлық ағындағы қысым түсуі мен шығын арасындағы аналитикалық және жартылай аналитикалық тәуелділіктерді сипаттайтын біртұтас математикалық негіз береді.

Түйін сөздер: Лежандр–Фенхель түрлендіруі; капиллярлық ағын; жалпыланған ньютондық сұйықтар; дөңес дуалдылық; ньютондық емес реология; қысыммен қозғалатын ағын; аққыштық шегі бар сұйықтар.

1 Introduction

Main provisions

A monotone flow curve of a generalized Newtonian fluid can be represented by a convex dissipation potential, while its Legendre-Fenchel conjugate provides a complementary potential in the stress variable. For fully developed capillary flow, the stress distribution is known from momentum balance; therefore, the velocity profile follows directly from the complementary potential without explicit inversion of the constitutive law. The framework recovers the Carreau-Yasuda solution and gives compact average-velocity relations for circular and planar geometries. The formulation also applies to Cross, Sisko, Ellis, Bingham–Papanastasiou and Herschel–Bulkley–Papanastasiou fluids, including ideal yield-stress limits through subdifferentials.

Pressure-driven flow in capillary geometries is a classical problem in fluid mechanics, transport phenomena, and rheology [1,2]. In its simplest form, the problem asks how a prescribed pressure gradient determines the velocity profile and the corresponding relation between flow rate and pressure drop. For a Newtonian fluid, the linear relation between shear stress and shear rate leads directly to the Hagen–Poiseuille solution. For a generalized Newtonian fluid, the same momentum balance remains simple, but the constitutive relation between shear stress and shear rate becomes nonlinear.

Capillary flow of non-Newtonian fluids has a long history in rheometry and polymer processing. Classical studies of plastic and non-Newtonian tube flow established the importance of pressure-drop/flow-rate relations, wall effects, and apparent viscosity corrections [3–6]. Modern treatments have further developed analytical, numerical, and semi-analytical descriptions of generalized Newtonian flow in pipes, slits, and narrow channels [7–11]. The present work is connected to this literature, but its purpose is more specific: it identifies a convex-duality structure underlying shear-rate-parametrized capillary-flow solutions.

The essential mathematical difficulty is the nonlinear constitutive relation. In fully developed capillary flow, the shear-stress distribution is determined explicitly by the momentum equation. In a

circular capillary of radius R , for example, the shear stress is a linear function of the radial coordinate. The shear rate is then obtained from the nonlinear flow curve

$$\tau = T(\dot{\gamma}), \quad (1)$$

where T maps the shear-rate magnitude to the shear-stress magnitude. Once $\dot{\gamma}$ is known as a function of position, the velocity follows by integration. Thus, the force balance gives the stress field directly; the remaining task is to invert, or otherwise represent, the nonlinear mapping from shear rate to stress.

In Ref. [12], this difficulty was avoided for the Carreau-Yasuda model by using the shear rate itself as the independent variable. In normalized variables, the stress balance gives the radial coordinate directly as a function of the normalized shear rate, and the velocity profile is then obtained in parametric form. This procedure led to closed expressions involving the Gauss hypergeometric function and provided benchmark solutions for numerical calculations. Related approaches have also used stress, shear rate, velocity, or flow rate as organizing variables for laminar tube-flow problems [9,10].

The present paper shows that this shear-rate parametrization can be placed in a broader Legendre-Fenchel transform framework. The Legendre-Fenchel transform is a standard construction in convex analysis [13,14]. For a monotone flow curve T , we define the dissipation potential

$$\mathcal{D}(\dot{\gamma}) = \int_0^{\dot{\gamma}} T(q) dq. \quad (2)$$

The constitutive relation between shear stress and shear rate is then the derivative, or more generally the subgradient, of a convex function. The Legendre-Fenchel transform of $\mathcal{D}$, denoted by $\mathcal{D}^*$, gives a complementary dissipation potential in the stress variable. In this dual representation, the inverse constitutive relation is encoded by the derivative or subgradient of $\mathcal{D}^*$.

This observation leads to a compact capillary-flow solution. Since the stress distribution is known explicitly, the velocity profile can be obtained by evaluating the complementary dissipation potential along that stress distribution. For a circular capillary with no slip at the wall, the result is

$$u(r) = \frac{R}{\tau_w} \left[\mathcal{D}^*(\tau_w) - \mathcal{D}^*\left(\tau_w \frac{r}{R}\right) \right], \quad (3)$$

where τ_w is the wall shear stress. Equation (3) is the central result of this work. It reformulates the capillary-flow problem as a convex-duality problem: the geometry determines the stress distribution, while the rheology enters through the complementary dissipation potential.

Unlike traditional capillary-flow analyses, which typically require a model-specific inversion of the constitutive relation, the present formulation reduces the problem to the construction of a complementary dissipation potential. This provides a unified treatment of smooth shear-thinning models, rheological laws naturally specified in inverse form, and nonsmooth ideal yield-stress limits within the same framework.

This formulation has several advantages. First, it separates the geometric part of the problem from the constitutive part. Second, it avoids the need to write the inverse constitutive relation explicitly when a parametric representation is sufficient. Third, it applies not only to smooth shear-thinning models, such as Carreau-Yasuda and Cross fluids [15–17], but also to regularized and ideal yield-stress models [3,18–20]. In the ideal yield-stress limit, the use of Legendre-Fenchel conjugacy is especially natural because nonsmooth constitutive behavior can be represented through subdifferentials rather than ordinary derivatives.

The analysis in this paper is restricted to steady, fully developed, isothermal, pressure-driven flow of monotone generalized Newtonian fluids in capillary geometries. Within this setting, the aim is to clarify the mathematical structure behind shear-rate-parametrized solutions and to show how several

common rheological models fit into the same framework. The paper is organized as follows. Section 2 recalls the Legendre-Fenchel transform ideas needed in this work and applies them to generalized Newtonian flow curves. Section 3 derives the general capillary-flow solution in terms of the complementary dissipation potential. Section 4 recovers the Carreau-Yasuda solution of Ref. [12]. Section 5 applies the same framework to other generalized Newtonian models, including Cross, Sisko, Ellis, Bingham-Papanastasiou, Herschel-Bulkley-Papanastasiou, and ideal yield-stress fluids. Section 6 summarizes the main findings and discusses the scope and possible extensions of the formulation.

2 Research methodology

Legendre-Fenchel transform and generalized Newtonian fluids

2.1 Definition and basic properties

The Legendre-Fenchel transform is a standard construction in convex analysis [13,14]. It is useful here because it provides a natural way to pass between a constitutive relation written in terms of shear rate and a complementary relation written in terms of stress.

To define the transform, it is convenient to allow functions that may take the value $+\infty$. Thus, instead of considering only real-valued functions, we consider functions of the form

$$f: X \rightarrow \mathbb{R} \cup \{+\infty\}. \quad (4)$$

Here X is the space of the variable x , while $f(x)$ is either a finite real number or $+\infty$. The value $+\infty$ is useful because it allows constraints to be included directly in the function. For example, if a variable is only allowed to lie in a set C , one can assign a finite value to $f(x)$ for $x \in C$ and assign $f(x) = +\infty$ for $x \notin C$. Points outside the allowed set are then automatically excluded from minimization problems.

In the present work, most calculations involve scalar variables such as shear-rate magnitude and shear-stress magnitude. Nevertheless, the extended-real notation is useful because it also covers constrained and nonsmooth cases, including ideal yield-stress fluids.

The Legendre-Fenchel transform of f , also called the convex conjugate of f , is defined by

$$f^*(p) = \sup_{x \in X} \{\langle p, x \rangle - f(x)\}. \quad (5)$$

Here x is the original variable, p is the dual variable, and $\langle p, x \rangle$ denotes the duality pairing. In finite-dimensional Euclidean space, this pairing is simply the dot product.

Equation (5) should be compared with the classical Legendre transform. If f is differentiable and strictly convex, one introduces the dual variable by

$$p = f'(x), \quad (6)$$

solves this relation for $x = x(p)$, and writes

$$f^*(p) = p x(p) - f(x(p)). \quad (7)$$

Thus, in the differentiable and strictly convex case, the Legendre-Fenchel transform reduces to the usual Legendre transform. The advantage of Eq. (5) is that it does not require differentiability or strict convexity. It is therefore the appropriate transform for nonsmooth functions, constrained functions, and yield-stress limits.

A useful way to understand the transform is through supporting lines. For a convex function, the affine function

$$x \mapsto \langle p, x \rangle - f^*(p) \quad (8)$$

lies below $f(x)$ for all x . The number $f^*(p)$ therefore determines the intercept of the best affine lower support with slope p . This is the geometric reason why the Legendre-Fenchel transform is naturally connected with convexity.

The fundamental inequality associated with Eq. (5) is the Fenchel-Young inequality,

$$f(x) + f^*(p) \geq \langle p, x \rangle. \quad (9)$$

Equality holds precisely when p is a subgradient of f at x :

$$p \in \partial f(x). \quad (10)$$

Equivalently,

$$x \in \partial f^*(p). \quad (11)$$

If both functions are differentiable, these conditions reduce to

$$p = f'(x), \quad x = (f^*)'(p). \quad (12)$$

Thus, the derivative of the transformed function gives the inverse of the derivative of the original function. This inverse relation is the key mathematical mechanism used in the capillary-flow problem below. Two comments are useful for the present application. First, the Legendre-Fenchel transform is always convex as a function of the dual variable. Second, the transform is not automatically self-inverse. The double transform recovers the original function when the original function is closed and convex; otherwise, it gives the closed convex envelope. In the present work, the relevant functions are dissipation potentials generated by monotone flow curves, so convexity is built into the formulation.

2.2 Dissipation and complementary dissipation potentials

We now apply the Legendre-Fenchel transform to generalized Newtonian fluids. Let $\dot{\gamma} \geq 0$ denote the shear-rate magnitude. A generalized Newtonian fluid is described by a scalar flow curve

$$\tau = T(\dot{\gamma}), \quad (13)$$

where $\tau \geq 0$ is the shear-stress magnitude. For a viscosity model $\eta(\dot{\gamma})$, the flow curve is

$$T(\dot{\gamma}) = \eta(\dot{\gamma})\dot{\gamma}. \quad (14)$$

We assume throughout that the flow curve is monotone nondecreasing:

$$T'(\dot{\gamma}) \geq 0. \quad (15)$$

For differentiable viscosity functions, this condition is equivalent to

$$\eta(\dot{\gamma}) + \dot{\gamma}\eta'(\dot{\gamma}) \geq 0. \quad (16)$$

Physically, monotonicity means that increasing the shear stress should not reduce the shear rate. Mathematically, it ensures that the flow curve can be generated by a convex potential. All model-specific formulas below are understood to hold for parameter ranges where this monotonicity condition is satisfied. Define the dissipation potential by

$$\mathcal{D}(\dot{\gamma}) = \int_0^{\dot{\gamma}} T(q) dq. \quad (17)$$

If T is differentiable, then

$$\mathcal{D}''(\dot{\gamma}) = T'(\dot{\gamma}) \geq 0, \quad (18)$$

so $\mathcal{D}$ is convex. The constitutive relation can therefore be written in the subgradient form

$$\tau \in \partial \mathcal{D}(\dot{\gamma}). \quad (19)$$

For smooth flow curves, this reduces to

$$\tau = \mathcal{D}'(\dot{\gamma}) = T(\dot{\gamma}). \quad (20)$$

The Legendre-Fenchel transform of $\mathcal{D}$ is

$$\mathcal{D}^*(\tau) = \sup_{\dot{\gamma} \geq 0} [\tau \dot{\gamma} - \mathcal{D}(\dot{\gamma})]. \quad (21)$$

We call $\mathcal{D}^*$ the complementary dissipation potential. The original potential $\mathcal{D}$ is written in terms of the shear-rate variable, while the complementary potential $\mathcal{D}^*$ is written in terms of the stress variable.

The inverse constitutive relation is obtained from the subgradient of $\mathcal{D}^*$:

$$\dot{\gamma} \in \partial \mathcal{D}^*(\tau). \quad (22)$$

For smooth strictly monotone flow curves, this becomes

$$\dot{\gamma} = (\mathcal{D}^*)'(\tau). \quad (23)$$

Thus, $\mathcal{D}$ gives stress as a function of shear rate, while $\mathcal{D}^*$ gives shear rate as a function of stress. In nonsmooth cases, such as ideal yield-stress limits, the derivative relation is replaced by the corresponding subgradient inclusion.

This is the point at which the Legendre-Fenchel transform becomes useful for capillary flow. The momentum equation gives the stress distribution directly, whereas the velocity gradient is determined by the shear rate. A formulation in which the shear rate is obtained from the stress variable is therefore well suited to the capillary-flow problem. The complementary dissipation potential provides this inverse constitutive information without requiring the inverse flow curve to be written explicitly in elementary form.

In the following section, this observation is used to obtain the velocity profile directly from $\mathcal{D}^*$.

3 Results of the study

Legendre-Fenchel solution of capillary flow

3.1 Circular capillary

Consider steady, fully developed, pressure-driven flow in a straight circular capillary of radius R . Let

$$G = \frac{P_0 - P_L}{L} > 0 \quad (24)$$

be the pressure drop per unit length. The axial velocity is

$$u = u(r), \quad (25)$$

and the shear-rate magnitude is

$$\dot{\gamma}(r) = -\frac{du}{dr} \geq 0. \quad (26)$$

The minus sign appears because the axial velocity decreases from the centerline to the wall.

The wall shear stress is

$$\tau_w = \frac{GR}{2}. \quad (27)$$

Momentum balance gives the linear stress distribution

$$\tau(r) = \tau_w \frac{r}{R}. \quad (28)$$

This relation is independent of the constitutive model. The rheological model enters only through the relation between τ and $\dot{\gamma}$.

The wall shear rate $\dot{\gamma}_w$ is defined by

$$T(\dot{\gamma}_w) = \tau_w. \quad (29)$$

For many models, this scalar equation must be solved numerically. For some models, such as power-law fluids and Bingham-Papanastasiou fluids, it can be solved explicitly. Throughout this section, we assume a monotone generalized Newtonian flow curve so that the stress–shear-rate relation can be represented by a convex dissipation potential.

3.2 Velocity profile from the complementary potential

Let $\mathcal{D}(\dot{\gamma})$ be the local dissipation potential and let $\mathcal{D}^*(\tau)$ be its Legendre-Fenchel transform,

$$\mathcal{D}^*(\tau) = \sup_{\dot{\gamma} \geq 0} [\tau \dot{\gamma} - \mathcal{D}(\dot{\gamma})]. \quad (30)$$

For a smooth strictly monotone generalized Newtonian fluid, the complementary constitutive relation is

$$\dot{\gamma} = \frac{d\mathcal{D}^*}{d\tau}. \quad (31)$$

In nonsmooth cases, this derivative relation should be replaced by the corresponding subgradient inclusion $\dot{\gamma} \in \partial \mathcal{D}^*(\tau)$.

Using Eq. (28), the velocity gradient can be written as

$$-\frac{du}{dr} = \frac{d\mathcal{D}^*}{d\tau} \left(\tau_w \frac{r}{R} \right) \quad (32)$$

in the smooth case. With the no-slip condition $u(R) = 0$, the velocity at radius r is obtained by integrating from the wall to the interior:

$$u(r) = \int_r^R \dot{\gamma}(s) \, ds. \quad (33)$$

Since

$$\tau = \tau_w \frac{s}{R}, \quad ds = \frac{R}{\tau_w} d\tau, \quad (34)$$

Eq. (33) becomes

$$u(r) = \frac{R}{\tau_w} \int_{\tau_w r/R}^{\tau_w} \frac{d\mathcal{D}^*}{d\tau}(\tau) \, d\tau. \quad (35)$$

Therefore,

$$u(r) = \frac{R}{\tau_w} \left[\mathcal{D}^*(\tau_w) - \mathcal{D}^*\left(\tau_w \frac{r}{R}\right) \right]. \quad (36)$$

Equation (36) is the central formula of this work. It states that the velocity profile is generated by the difference between the complementary dissipation potential evaluated at the wall stress and at the local stress. The geometry enters through the known stress distribution in Eq. (28), while the rheology

enters through the complementary potential $\mathcal{D}^*(\tau)$. The formula is derived here for no-slip circular capillary flow, but the same duality structure will also appear in the dimensionless and planar formulations below.

3.3 Dimensionless formulation

It is useful to rewrite the solution in dimensionless form. Let

$$s = \frac{\dot{\gamma}}{\dot{\gamma}_w}, \quad \rho = \frac{r}{R} = \frac{\tau}{\tau_w}, \quad (37)$$

where $\dot{\gamma}_w$ is the wall shear rate and τ_w is the wall shear stress. Define the normalized flow curve

$$\Psi(s) = \frac{T(\dot{\gamma}_w s)}{\tau_w}. \quad (38)$$

Because $T(\dot{\gamma}_w) = \tau_w$, one has

$$\Psi(1) = 1. \quad (39)$$

The stress distribution gives

$$\rho = \Psi(s). \quad (40)$$

Thus, the dimensionless radial coordinate ρ is the normalized stress corresponding to the dimensionless shear rate s . Under the monotonicity assumption on T , the function $\Psi(s)$ is monotone nondecreasing and can be used as a parametric representation of the radial coordinate.

The normalized dissipation potential is

$$\Phi(s) = \frac{\mathcal{D}(\dot{\gamma}_w s)}{\tau_w \dot{\gamma}_w}. \quad (41)$$

Using Eq. (17), this can be written as

$$\Phi(s) = \int_0^s \Psi(\xi) \, d\xi. \quad (42)$$

Therefore,

$$\Phi'(s) = \Psi(s) = \rho. \quad (43)$$

This shows that ρ is the conjugate variable to the normalized shear rate s .

The corresponding normalized complementary potential is

$$\Phi^*(\rho) = \frac{\mathcal{D}^*(\tau_w \rho)}{\tau_w \dot{\gamma}_w}. \quad (44)$$

Equivalently,

$$\Phi^*(\rho) = \sup_{s \geq 0} [\rho s - \Phi(s)]. \quad (45)$$

The dimensionless velocity is defined as

$$\tilde{u}(\rho) = \frac{u(r)}{R \dot{\gamma}_w}. \quad (46)$$

Using Eq. (36), one obtains

$$\tilde{u}(\rho) = \Phi^*(1) - \Phi^*(\rho). \quad (47)$$

If $\rho = \Psi(s)$, then the Fenchel equality gives

$$\Phi^*(\rho) = s\rho - \Phi(s) = s\Psi(s) - \Phi(s). \quad (48)$$

At the wall, $s = 1$ and $\rho = 1$. Since $\Psi(1) = 1$,

$$\Phi^*(1) = 1 - \Phi(1). \quad (49)$$

Therefore, the velocity profile can also be written in shear-rate-parametrized form as

$$\tilde{u}(s) = 1 - \Phi(1) - s\Psi(s) + \Phi(s). \quad (50)$$

Using Eq. (42), this becomes

$$\tilde{u}(s) = 1 - s\Psi(s) - \int_s^1 \Psi(\xi) d\xi. \quad (51)$$

Equation (51) is the general shear-rate-parametrized form of the velocity profile. It is equivalent to the construction used in Ref. [12], where $\Psi(s)$ was written as the dimensionless radial coordinate. The auxiliary function introduced there is the tail integral

$$w(s) = \int_s^1 \Psi(\xi) d\xi. \quad (52)$$

Thus, the earlier shear-rate-parametrized solution can be interpreted as a Legendre-Fenchel transform calculation in normalized variables.

3.4 Average velocity

The dimensionless average velocity for circular capillary flow is obtained by area averaging the dimensionless velocity profile:

$$\bar{\tilde{u}} = 2 \int_0^1 \tilde{u}(\rho) \rho d\rho = \int_0^1 \tilde{u}(\rho) d(\rho^2). \quad (53)$$

Using integration by parts, together with the no-slip condition $\tilde{u}(1) = 0$, gives

$$\bar{\tilde{u}} = - \int_0^1 \rho^2 d\tilde{u}. \quad (54)$$

From Eq. (51),

$$\frac{d\tilde{u}}{d\rho} = -s. \quad (55)$$

Therefore,

$$\bar{\tilde{u}} = \int_0^1 \rho^2 s d\rho. \quad (56)$$

Since $\rho = \Psi(s)$, a second integration by parts gives

$$\bar{\tilde{u}} = \frac{1}{3} \left[1 - \int_0^1 \rho(s)^3 ds \right]. \quad (57)$$

Thus, for circular capillary flow,

$$\bar{\tilde{u}} = \frac{1}{3} \left[1 - \int_0^1 \Psi(s)^3 ds \right]. \quad (58)$$

The same calculation also applies to planar Poiseuille flow, with the area weighting replaced by the one-dimensional average over half of the channel. Using symmetry about the center plane, one has

$$\bar{\tilde{u}} = \int_0^1 \tilde{u}(\rho) d\rho. \quad (59)$$

Integration by parts gives

$$\bar{\tilde{u}} = \int_0^1 \rho s d\rho = \frac{1}{2} \left[1 - \int_0^1 \rho(s)^2 ds \right]. \quad (60)$$

Hence, for planar Poiseuille flow,

$$\bar{\tilde{u}} = \frac{1}{2} \left[1 - \int_0^1 \Psi(s)^2 ds \right]. \quad (61)$$

Both results can be written in a unified form by introducing the geometric index

$$d = \begin{cases} 1, & \text{planar Poiseuille flow,} \\ 2, & \text{circular capillary flow.} \end{cases} \quad (62)$$

The dimensionless average velocity is then

$$\bar{u} = \frac{1}{d+1} \left[1 - \int_0^1 \Psi(s)^{d+1} ds \right]. \quad (63)$$

For circular capillary flow, the dimensional volumetric flow rate is

$$Q = \pi R^2 \bar{u} = \pi R^3 \dot{\gamma}_w \bar{u}. \quad (64)$$

Therefore,

$$Q = \frac{\pi R^3 \dot{\gamma}_w}{3} \left[1 - \int_0^1 \Psi(s)^3 ds \right]. \quad (65)$$

Together with

$$\tau_w = \frac{GR}{2}, \quad \Psi(s) = \frac{T(\dot{\gamma}_w s)}{\tau_w}, \quad \Psi(1) = 1, \quad (66)$$

Eq. (65) gives the flow-rate–pressure-drop relation once the rheological model $T(\dot{\gamma})$ is specified.

4 Recovery of the Carreau-Yasuda result

We now show how the general Legendre-Fenchel formulation recovers the Carreau-Yasuda capillary-flow result. The Carreau-Yasuda viscosity model is

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty) [1 + (\lambda \dot{\gamma})^a]^{(n-1)/a}, \quad (67)$$

where η_0 is the zero-shear viscosity, η_∞ is the infinite-shear viscosity, λ is a characteristic time scale, a is the Yasuda transition parameter, and n is the power-law index [15,16]. The corresponding flow curve is

$$T(\dot{\gamma}) = \eta(\dot{\gamma}) \dot{\gamma}. \quad (68)$$

The following derivation assumes parameter ranges and shear-rate intervals for which this flow curve is monotone nondecreasing, so that the associated dissipation potential is convex.

Let $\dot{\gamma}_w$ be the wall shear rate. We define

$$\tilde{\lambda} = \lambda \dot{\gamma}_w, \quad \eta_w = \frac{\tau_w}{\dot{\gamma}_w} = \eta(\dot{\gamma}_w). \quad (69)$$

The normalized viscosities are

$$\tilde{\eta}_0 = \frac{\eta_0}{\eta_w}, \quad \tilde{\eta}_\infty = \frac{\eta_\infty}{\eta_w}, \quad \Delta = \tilde{\eta}_0 - \tilde{\eta}_\infty. \quad (70)$$

Therefore,

$$\tilde{\eta}(s) = \frac{\eta(\dot{\gamma}_w s)}{\eta_w} = \tilde{\eta}_\infty + \Delta [1 + (\tilde{\lambda} s)^a]^{(n-1)/a}. \quad (71)$$

The wall normalization gives

$$\tilde{\eta}(1) = 1, \quad \tilde{\eta}_\infty + \Delta [1 + \tilde{\lambda}^a]^{(n-1)/a} = 1. \quad (72)$$

The normalized flow curve is

$$\Psi(s) = \frac{T(\dot{\gamma}_w s)}{\tau_w} = s\tilde{\eta}(s). \quad (73)$$

Thus,

$$\Psi(s) = s \left\{ \tilde{\eta}_\infty + \Delta [1 + (\tilde{\lambda}s)^a]^{(n-1)/a} \right\}. \quad (74)$$

At the wall, $s = 1$ and $\rho = 1$, so that

$$\Psi(1) = 1. \quad (75)$$

The normalized dissipation potential is

$$\Phi(s) = \int_0^s \Psi(\xi) d\xi. \quad (76)$$

For the Carreau-Yasuda model, this integral contains terms of the form

$$\int_0^s \xi [1 + (\tilde{\lambda}\xi)^a]^{(n-1)/a} d\xi. \quad (77)$$

This integral is generally not elementary. It can be expressed in terms of the Gauss hypergeometric function, denoted by ${}_2F_1$ [21]. We use the standard integral identity

$$\int_0^s x^{p-1} (1 + \beta x^a)^q dx = \frac{s^p}{p} {}_2F_1\left(\frac{p}{a}, -q; 1 + \frac{p}{a}; -\beta s^a\right), \quad (78)$$

provided the integral is finite, with analytic continuation understood where applicable [21]. Applying Eq. (78) with

$$p = 2, \quad \beta = \tilde{\lambda}^a, \quad q = \frac{n-1}{a}, \quad (79)$$

gives

$$\Phi(s) = \frac{1}{2} \tilde{\eta}_\infty s^2 + \frac{1}{2} \Delta s^2 {}_2F_1\left(\frac{2}{a}, \frac{1-n}{a}; \frac{a+2}{a}; -(\tilde{\lambda}s)^a\right). \quad (80)$$

The velocity profile follows from the general dimensionless formula. Since $\rho = \Psi(s)$, one has

$$\rho(s) = \Psi(s). \quad (81)$$

The shear-rate-parametrized dimensionless velocity is

$$\tilde{u}(s) = 1 - s\Psi(s) - \int_s^1 \Psi(\xi) d\xi. \quad (82)$$

Using $\Phi(s) = \int_0^s \Psi(\xi) d\xi$, this becomes

$$\tilde{u}(s) = 1 - s\Psi(s) - \Phi(1) + \Phi(s). \quad (83)$$

Therefore, the Carreau-Yasuda velocity profile can be written in the closed parametric form

$$\rho(s) = \Psi(s), \quad \tilde{u}(s) = 1 - s\Psi(s) - \Phi(1) + \Phi(s), \quad (84)$$

where $\Psi(s)$ is given by Eq. (74) and $\Phi(s)$ is given by Eq. (80). This is the Legendre-Fenchel reformulation of the shear-rate-parametrized velocity solution of Ref. [12].

The average velocity follows from the unified formula derived above. Define, for integer $k \geq 1$, the moment

$$M_k = \int_0^1 \Psi(s)^k ds. \quad (85)$$

For the Carreau-Yasuda model,

$$\Psi(s)^k = s^k \left\{ \tilde{\eta}_\infty + \Delta [1 + (\tilde{\lambda}s)^a]^{(n-1)/a} \right\}^k. \quad (86)$$

Expanding the power gives

$$M_k = \sum_{j=0}^k \binom{k}{j} \tilde{\eta}_\infty^{k-j} \Delta^j \int_0^1 s^k [1 + (\tilde{\lambda}s)^a]^{j(n-1)/a} ds. \quad (87)$$

Using Eq. (78) with

$$p = k + 1, \quad \beta = \tilde{\lambda}^a, \quad q = \frac{j(n-1)}{a}, \quad (88)$$

one obtains

$$M_k = \frac{1}{k+1} \sum_{j=0}^k \binom{k}{j} \tilde{\eta}_\infty^{k-j} \Delta^j \times {}_2F_1\left(\frac{k+1}{a}, \frac{j(1-n)}{a}; \frac{k+1+a}{a}; -\tilde{\lambda}^a\right). \quad (89)$$

The dimensionless average velocity is therefore

$$\bar{u} = \frac{1}{d+1} [1 - M_{d+1}]. \quad (90)$$

For circular capillary flow, $d = 2$ and $k = 3$; for planar Poiseuille flow, $d = 1$ and $k = 2$. Thus,

$$\bar{u}_{\text{pipe}} = \frac{1}{3} [1 - M_3], \quad \bar{u}_{\text{plane}} = \frac{1}{2} [1 - M_2]. \quad (91)$$

For circular capillary flow, the dimensional volumetric flow rate is

$$Q = \pi R^2 \bar{u} = \pi R^3 \dot{\gamma}_w \bar{u}_{\text{pipe}}. \quad (92)$$

Thus,

$$Q = \frac{\pi R^3 \dot{\gamma}_w}{3} [1 - M_3]. \quad (93)$$

The pressure-gradient dependence enters through

$$\tau_w = \frac{GR}{2} = \eta(\dot{\gamma}_w) \dot{\gamma}_w. \quad (94)$$

For a prescribed pressure gradient G , Eq. (94) determines $\dot{\gamma}_w$, and Eq. (93) gives the corresponding volumetric flow rate.

Equations (84) and (90) recover the pipe-flow and plane-flow Carreau-Yasuda results of Ref. [12]. The present derivation also clarifies why hypergeometric functions appear: they are consequences of analytically evaluating the normalized dissipation potential and the moments of the normalized flow curve, rather than essential ingredients of the Legendre-Fenchel framework. The underlying conjugacy structure,

$$s \leftrightarrow \rho = \Psi(s), \quad \tilde{u}(\rho) = \Phi^*(1) - \Phi^*(\rho), \quad (95)$$

is independent of the particular viscosity model. The Carreau-Yasuda model is therefore one explicit example within a broader Legendre-Fenchel framework for pressure-driven capillary flow.

5 Other generalized Newtonian models

The preceding sections show that, once the normalized flow curve $\Psi(s) = T(\dot{\gamma}_w s)/\tau_w$ is known, the velocity profile and the average velocity follow from Eqs. (51) and (63). This section records several representative examples. The purpose is not to rederive each tube-flow solution in detail, but to show how different rheological models enter the same Legendre-Fenchel structure through their normalized flow curves, dissipation potentials, and moment integrals.

For elementary models, the framework recovers the classical results directly. For empirical shear-thinning models, such as Sisko, Cross, and Carreau-Yasuda fluids, the same construction gives parametric velocity profiles and average-velocity moments. For models naturally written in inverse form, such as the Ellis model, the complementary potential can be obtained directly. For regularized and ideal yield-stress fluids, the formulation is especially useful because it also handles nonsmooth limits through subgradients.

5.1 Newtonian and power-law limits

For a Newtonian fluid, $T(\dot{\gamma}) = \eta\dot{\gamma}$, and the complementary potential is $\mathcal{D}^*(\tau) = \tau^2/(2\eta)$. Substitution into Eq. (36) gives the classical Hagen–Poiseuille profile,

$$u(r) = \frac{G}{4\eta} (R^2 - r^2). \quad (96)$$

In dimensionless form, $\Psi(s) = s$, and $\tilde{u} = 1/4$ for circular capillary flow.

For a power-law fluid,

$$T(\dot{\gamma}) = K\dot{\gamma}^n, \quad n > 0, \quad (97)$$

the normalized flow curve is $\Psi(s) = s^n$. Therefore, $\rho = s^n$, and the dimensionless velocity profile is

$$\tilde{u}(\rho) = \frac{n}{n+1} [1 - \rho^{(n+1)/n}]. \quad (98)$$

The corresponding average velocities are

$$\tilde{u} = \frac{n}{3n+1}, \text{circular capillary flow}, \quad (99)$$

$$\tilde{u} = \frac{n}{2n+1}, \text{planar Poiseuille flow}. \quad (100)$$

These elementary limits serve as consistency checks on the general formula.

5.2 Sisko fluid

The Sisko model combines a Newtonian high-shear contribution with a power-law contribution:

$$T(\dot{\gamma}) = \eta_\infty \dot{\gamma} + K\dot{\gamma}^n. \quad (101)$$

Define

$$\mu = \frac{\eta_\infty \dot{\gamma}_w}{\tau_w}, \quad \kappa = \frac{K \dot{\gamma}_w^n}{\tau_w}. \quad (102)$$

The wall condition gives $\mu + \kappa = 1$. The normalized flow curve and normalized dissipation potential are

$$\Psi(s) = \mu s + \kappa s^n, \quad \Phi(s) = \frac{\mu}{2} s^2 + \frac{\kappa}{n+1} s^{n+1}. \quad (103)$$

Therefore,

$$\tilde{u}(s) = \frac{\mu}{2}(1 - s^2) + \frac{n\kappa}{n+1}(1 - s^{n+1}). \quad (104)$$

For integer k , the moment integral is elementary:

$$M_k = \sum_{j=0}^k \binom{k}{j} \mu^{k-j} \kappa^j \frac{1}{k-j+nj+1}. \quad (105)$$

The average velocity follows from Eq. (63) with M_{d+1} .

5.3 Cross fluid

A common Cross-type viscosity model is

$$\eta(\dot{\gamma}) = \eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + (\lambda\dot{\gamma})^m}. \quad (106)$$

This model is widely used for pseudoplastic and polymeric fluids [17]. As pointed out in Ref. [12], it can also be viewed as a special case of the Carreau-Yasuda form. Comparing Eq. (106) with Eq. (67), the Cross model is recovered by the formal parameter choice

$$a = m, \quad n = 1 - m. \quad (107)$$

This mapping gives $(n - 1)/a = -1$, and therefore reduces the Carreau-Yasuda factor to $[1 + (\lambda\dot{\gamma})^m]^{-1}$. It is a formal mapping at the level of the viscosity function and should not be confused with independent empirical parameter conventions used in some Cross-model literature.

Let

$$\tilde{\lambda} = \lambda\dot{\gamma}_w, \quad \eta_w = \frac{\tau_w}{\dot{\gamma}_w}, \quad (108)$$

and define

$$\tilde{\eta}_\infty = \frac{\eta_\infty}{\eta_w}, \quad \Delta = \frac{\eta_0 - \eta_\infty}{\eta_w}. \quad (109)$$

The normalized flow curve is

$$\Psi(s) = s \left[\tilde{\eta}_\infty + \frac{\Delta}{1 + (\tilde{\lambda}s)^m} \right], \quad (110)$$

with wall normalization

$$\tilde{\eta}_\infty + \frac{\Delta}{1 + \tilde{\lambda}^m} = 1. \quad (111)$$

Using the hypergeometric integral identity introduced in Sec. 4, the normalized dissipation potential is

$$\begin{aligned} \Phi(s) &= \frac{1}{2} \tilde{\eta}_\infty s^2 \\ &+ \frac{1}{2} \Delta s^2 {}_2F_1\left(\frac{2}{m}, 1; \frac{m+2}{m}; -(\tilde{\lambda}s)^m\right). \end{aligned} \quad (112)$$

The velocity profile is then given by Eq. (50). For integer k , define $M_k = \int_0^1 \Psi(s)^k ds$. Expanding the integrand and using the same hypergeometric identity gives

$$\begin{aligned} M_k &= \frac{1}{k+1} \sum_{j=0}^k \binom{k}{j} \tilde{\eta}_\infty^{k-j} \Delta^j \\ &\times {}_2F_1\left(\frac{k+1}{m}, j; \frac{k+1+m}{m}; -\tilde{\lambda}^m\right). \end{aligned} \quad (113)$$

The average velocity follows from Eq. (63). Thus, the Cross model gives the same type of hypergeometric structure as the Carreau-Yasuda model, but with the exponent fixed at -1 .

5.4 Ellis fluid

The Ellis model is particularly illustrative from the Legendre-Fenchel viewpoint because it is naturally specified in the dual stress variable. As a result, the complementary potential can be obtained directly without first inverting the constitutive relation.

The Ellis model is often written in inverse-flow-curve form:

$$\dot{\gamma}(\tau) = \frac{\tau}{\eta_0} \left[1 + \left(\frac{\tau}{\tau_{1/2}} \right)^{\alpha-1} \right]. \quad (114)$$

This form is especially natural in the present framework because the complementary potential is obtained directly by integration in the stress variable:

$$\mathcal{D}^*(\tau) = \int_0^\tau \dot{\gamma}(q) dq = \frac{\tau^2}{2\eta_0} + \frac{\tau^{\alpha+1}}{(\alpha+1)\eta_0\tau_{1/2}^{\alpha-1}}. \quad (115)$$

Substitution into Eq. (36) gives

$$u(r) = \frac{R\tau_w}{2\eta_0} (1 - \rho^2) + \frac{R\tau_w^\alpha}{(\alpha+1)\eta_0\tau_{1/2}^{\alpha-1}} (1 - \rho^{\alpha+1}), \quad \rho = \frac{r}{R}. \quad (116)$$

This example highlights an advantage of the conjugacy viewpoint: if a model is naturally specified as $\dot{\gamma}(\tau)$, the complementary potential can be obtained without inverting the constitutive law.

5.5 Bingham–Papanastasiou fluid

Yield-stress fluids provide a useful test case because ideal yield-stress laws are nonsmooth, while Papanastasiou regularization gives a smooth approximation that approaches the ideal law as the regularization parameter increases [18]. For a Bingham–Papanastasiou fluid,

$$T(\dot{\gamma}) = \eta_p \dot{\gamma} + \tau_y (1 - e^{-m\dot{\gamma}}), \quad (117)$$

where η_p is the plastic viscosity, τ_y is the yield stress, and m is the regularization parameter. The corresponding dissipation potential is

$$\mathcal{D}(\dot{\gamma}) = \frac{1}{2}\eta_p \dot{\gamma}^2 + \tau_y \left[\dot{\gamma} - \frac{1-e^{-m\dot{\gamma}}}{m} \right], \quad (118)$$

which is smooth and strictly convex for positive material parameters.

The wall shear rate satisfies

$$\tau_w = \eta_p \dot{\gamma}_w + \tau_y (1 - e^{-m\dot{\gamma}_w}). \quad (119)$$

This scalar equation can be solved explicitly using the principal branch W_0 of the Lambert W function, defined by $W(z)e^{W(z)} = z$ [22]. The use of the Lambert W function in Papanastasiou and related non-Newtonian flow problems has been discussed in Refs. [23–25]. For the present wall-shear-rate problem, the result is

$$\dot{\gamma}_w = \frac{\tau_w - \tau_y}{\eta_p} + \frac{1}{m} W_0 \left[\frac{m\tau_y}{\eta_p} \exp \left(-\frac{m(\tau_w - \tau_y)}{\eta_p} \right) \right]. \quad (120)$$

For positive material parameters, the argument of W_0 is positive, and the principal real branch gives the physical solution.

Define

$$\beta = \frac{\tau_y}{\tau_w}, \quad \mu = \frac{\eta_p \dot{\gamma}_w}{\tau_w}, \quad M = m \dot{\gamma}_w. \quad (121)$$

The wall condition is

$$\mu + \beta(1 - e^{-M}) = 1. \quad (122)$$

The normalized flow curve and normalized dissipation potential are

$$\begin{aligned} \Psi(s) &= \mu s + \beta(1 - e^{-Ms}), \\ \Phi(s) &= \frac{\mu}{2} s^2 + \beta \left[s - \frac{1 - e^{-Ms}}{M} \right]. \end{aligned} \quad (123)$$

The parametric velocity profile is

$$\begin{aligned} \tilde{u}(s) &= \frac{\mu}{2} (1 - s^2) + \beta (s e^{-Ms} - e^{-M}) \\ &\quad + \frac{\beta}{M} (e^{-Ms} - e^{-M}). \end{aligned} \quad (124)$$

The map $\rho = \Psi(s)$ can also be inverted explicitly:

$$\begin{aligned} s(\rho) &= \frac{\rho - \beta}{\mu} \\ &\quad + \frac{1}{M} W_0 \left[\frac{M\beta}{\mu} \exp \left(-\frac{M(\rho - \beta)}{\mu} \right) \right]. \end{aligned} \quad (125)$$

For circular capillary flow, the average velocity is $\tilde{u} = (1 - M_3)/3$, where

$$M_3 = \int_0^1 [\mu s + \beta(1 - e^{-Ms})]^3 ds. \quad (126)$$

This integral is elementary. A compact closed form is

$$\begin{aligned} M_3 &= \beta^3 + \frac{3}{2} \beta^2 \mu + \beta \mu^2 + \frac{\mu^3}{4} \\ &\quad - 3\beta [\beta^2 A_0(M) + 2\beta \mu A_1(M) + \mu^2 A_2(M)] \\ &\quad + 3\beta^2 [\beta A_0(2M) + \mu A_1(2M)] \\ &\quad - \beta^3 A_0(3M), \end{aligned} \quad (127)$$

Where

$$\begin{aligned} A_0(x) &= \frac{1 - e^{-x}}{x}, \\ A_1(x) &= \frac{1 - e^{-x}(1+x)}{x^2}, \\ A_2(x) &= \frac{2 - e^{-x}(x^2 + 2x + 2)}{x^3}. \end{aligned} \quad (128)$$

The dimensional flow rate is then

$$Q = \pi R^3 \dot{\gamma}_w \frac{1 - M_3}{3}. \quad (129)$$

5.6 Herschel–Bulkley–Papanastasiou fluid

The Herschel–Bulkley–Papanastasiou model generalizes the Bingham–Papanastasiou model by replacing the linear viscous contribution with a power-law contribution:

$$T(\dot{\gamma}) = K\dot{\gamma}^n + \tau_y(1 - e^{-m\dot{\gamma}}). \quad (130)$$

For $n = 1$ and $K = \eta_p$, Eq. (130) reduces to Eq. (117). The corresponding dissipation potential is

$$\mathcal{D}(\dot{\gamma}) = \frac{K}{n+1} \dot{\gamma}^{n+1} + \tau_y \left[\dot{\gamma} - \frac{1-e^{-m\dot{\gamma}}}{m} \right]. \quad (131)$$

For $K > 0$, $n > 0$, $\tau_y \geq 0$, and $m > 0$, the flow curve is monotone nondecreasing and the dissipation potential is convex on $\dot{\gamma} \geq 0$. When $0 < n < 1$, the derivative of the flow curve is singular as $\dot{\gamma} \rightarrow 0^+$, but this singularity is compatible with convexity.

The wall shear rate is determined by

$$\tau_w = K\dot{\gamma}_w^n + \tau_y(1 - e^{-m\dot{\gamma}_w}). \quad (132)$$

Unlike the Bingham–Papanastasiou case, this equation generally cannot be reduced to the standard Lambert W form when $n \neq 1$, because it contains both a power-law term and an exponential term. It is nevertheless a scalar monotone equation.

Define

$$\alpha = \frac{K\dot{\gamma}_w^n}{\tau_w}, \quad \beta = \frac{\tau_y}{\tau_w}, \quad M = m\dot{\gamma}_w. \quad (133)$$

The wall condition is

$$\alpha + \beta(1 - e^{-M}) = 1. \quad (134)$$

The normalized flow curve and normalized dissipation potential are

$$\begin{aligned} \Psi(s) &= \alpha s^n + \beta(1 - e^{-Ms}), \\ \Phi(s) &= \frac{\alpha}{n+1} s^{n+1} + \beta \left[s - \frac{1-e^{-Ms}}{M} \right]. \end{aligned} \quad (135)$$

The parametric radial coordinate is $\rho(s) = \Psi(s)$. The corresponding velocity profile is

$$\begin{aligned} \tilde{u}(s) &= \frac{\alpha n}{n+1} (1 - s^{n+1}) \\ &+ \beta(s e^{-Ms} - e^{-M}) + \frac{\beta}{M} (e^{-Ms} - e^{-M}). \end{aligned} \quad (136)$$

The average-velocity moment is

$$M_k = \int_0^1 [\alpha s^n + \beta(1 - e^{-Ms})]^k ds. \quad (137)$$

For integer k , expansion gives

$$\begin{aligned} M_k &= \sum_{j=0}^k \binom{k}{j} \alpha^{k-j} \beta^j \\ &\times \sum_{\ell=0}^j (-1)^\ell \binom{j}{\ell} E_{n(k-j)}(\ell M), \end{aligned} \quad (138)$$

where

$$E_p(x) = \int_0^1 s^p e^{-xs} ds. \quad (139)$$

For $x > 0$,

$$E_p(x) = \frac{\gamma(p+1, x)}{x^{p+1}}, \quad (140)$$

where $\gamma(\cdot, \cdot)$ is the lower incomplete gamma function [21]. For $x = 0$, $E_p(0) = 1/(p+1)$. The dimensionless average velocity follows from Eq. (63). For circular capillary flow,

$$Q = \pi R^3 \dot{\gamma}_w \frac{1-M_3}{3}. \quad (141)$$

The Bingham–Papanastasiou result is recovered by setting $n = 1$ and $K = \eta_p$. Then $\alpha = \mu$, and Eqs. (135) and (136) reduce to Eqs. (123) and (124). In the same limit, Eq. (132) reduces to Eq. (119), and the explicit W_0 solution in Eq. (120) is recovered.

5.7 Ideal yield-stress limit

The Legendre–Fenchel formulation is especially useful in the ideal yield-stress limit. Consider $m \rightarrow \infty$ in the Herschel–Bulkley–Papanastasiou model. The regularized flow curve tends to the ideal Herschel–Bulkley law

$$T(\dot{\gamma}) = \tau_y + K\dot{\gamma}^n, \quad \dot{\gamma} > 0, \quad (142)$$

a classical yield-stress model [19,20]. At $\dot{\gamma} = 0$, the ideal law is not single-valued: stresses below the yield stress correspond to zero shear rate.

Let $\beta = \tau_y/\tau_w$ and assume $0 < \beta < 1$, so that yielding occurs near the wall. The wall shear rate satisfies

$$K\dot{\gamma}_w^n = \tau_w(1 - \beta). \quad (143)$$

In the sheared region,

$$\rho = \Psi(s) = \beta + (1 - \beta)s^n, \quad 0 < s \leq 1. \quad (144)$$

Thus, $s \rightarrow 0^+$ corresponds to $\rho \rightarrow \beta$, and the interval $0 \leq \rho \leq \beta$ is an unsheared plug.

The normalized ideal dissipation potential is

$$\Phi_{HB}(s) = \beta s + \frac{1-\beta}{n+1} s^{n+1}, \quad s \geq 0. \quad (145)$$

For the nonnegative stress interval relevant here, the physically relevant subgradient at $s = 0$ is $0 \leq \rho \leq \beta$. To verify the corresponding conjugate, we compute

$$\Phi_{HB}^*(\rho) = \sup_{s \geq 0} [\rho s - \beta s - \frac{1-\beta}{n+1} s^{n+1}]$$

For $0 \leq \rho \leq \beta$, the supremum is attained at $s = 0$, giving $\Phi_{HB}^*(\rho) = 0$. For $\rho > \beta$, the maximizer satisfies

$$\rho - \beta - (1 - \beta)s^n = 0,$$

so that

$$s^* = \left(\frac{\rho - \beta}{1 - \beta} \right)^{\frac{1}{n}}.$$

Substituting s^* into the supremum yields

$$\Phi_{HB}^*(\rho) = \begin{cases} 0, & 0 \leq \rho \leq \beta, \\ \frac{n}{n+1} \frac{(\rho - \beta)^{(n+1)/n}}{(1 - \beta)^{1/n}}, & \beta < \rho \leq 1. \end{cases} \quad (146)$$

The dimensionless velocity is therefore

$$\tilde{u}(\rho) = \Phi_{HB}^*(1) - \Phi_{HB}^*(\rho). \quad (147)$$

Since $\Phi_{HB}^*(1) = n(1 - \beta)/(n + 1)$, the plug velocity is

$$\tilde{u}(\rho) = \frac{n}{n+1} (1 - \beta), \quad 0 \leq \rho \leq \beta. \quad (148)$$

In the sheared region,

$$\begin{aligned} \tilde{u}(\rho) &= \frac{n}{n+1}(1-\beta) \\ &- \frac{n}{n+1} \frac{(\rho-\beta)^{(n+1)/n}}{(1-\beta)^{1/n}}, \quad \beta < \rho \leq 1. \end{aligned} \quad (149)$$

The average velocity follows from the same moment formula. For integer k ,

$$\begin{aligned} M_k &= \int_0^1 [\beta + (1-\beta)s^n]^k ds \\ &= \sum_{j=0}^k \binom{k}{j} \beta^{k-j} (1-\beta)^j \frac{1}{nj+1}. \end{aligned} \quad (150)$$

Although the integral over s runs only over the sheared branch, the plug contribution is included through the boundary term in the integration-by-parts derivation of the average velocity. Hence,

$$\bar{u} = \frac{1-M_{d+1}}{d+1}. \quad (151)$$

For the ideal Bingham case, $n = 1$. The plug velocity becomes

$$\tilde{u}(\rho) = \frac{1-\beta}{2}, \quad 0 \leq \rho \leq \beta, \quad (152)$$

and the sheared-region velocity becomes

$$\tilde{u}(\rho) = \frac{1-\beta}{2} - \frac{(\rho-\beta)^2}{2(1-\beta)}, \quad \beta < \rho \leq 1. \quad (153)$$

Thus, the plug region is not imposed separately. It follows directly from the subgradient of the nonsmooth dissipation potential, or equivalently from the flat region of the complementary potential. Plug formation is therefore the spatial manifestation of the dual variational structure of an ideal yield-stress fluid.

6 Discussion

The present work reformulates pressure-driven capillary flow of generalized Newtonian fluids using Legendre-Fenchel conjugacy. The usual difficulty in such problems is the implicit nonlinear relation between shear stress and shear rate. Although the stress distribution is known from momentum balance, the shear rate must still be determined from the nonlinear flow curve $\tau = T(\dot{\gamma})$. For many rheological models, this inversion cannot be performed in elementary functions. The shear-rate parametrization used in Ref. [12] avoided this difficulty by using the shear rate itself as the independent variable. The present manuscript shows that this parametrization is not merely a useful change of variables. It is a manifestation of the conjugate-variable structure generated by the dissipation potential.

In this interpretation, the capillary-flow problem has three distinct parts. The first is geometric: the stress distribution is fixed by momentum balance, as in Eq. (28). The second is constitutive: the flow curve defines the dissipation potential, as in Eq. (17). The third is dual: the velocity profile is generated by the complementary dissipation potential, as shown in Eq. (36). This decomposition separates the known stress field from the rheological response and explains why the same mathematical structure applies to apparently different viscosity models.

The representative models examined in this work are summarized in Table 1. The table is not intended to be an exhaustive catalog of generalized Newtonian fluids. Rather, it illustrates how the same Legendre-Fenchel construction applies to several common models, including Newtonian, power-law, Sisko, Cross, Carreau–Yasuda, Ellis, Bingham–Papanastasiou, Herschel–Bulkley–Papanastasiou, and ideal Herschel–Bulkley fluids. It also emphasizes that the central object in the

dimensionless formulation is the normalized flow curve $\Psi(s)$. Once $\Psi(s)$ is known, the velocity profile and average velocity follow from Eqs. (51) and (63).

Table 1. Representative model catalog in the Legendre-Fenchel capillary-flow framework. The normalized flow curve $\Psi(s) = T(\dot{\gamma}_w s)/\tau_w$ is the central object. The velocity and average velocity follow from Eqs. (51) and (63), respectively.

<i>Model</i>	<i>Flow curve or inverse law</i>	<i>Normalized flow curve</i>	<i>Analytical structure</i>
<i>Newtonian</i>	$T = \eta \dot{\gamma}$	$\Psi = s$	<i>elementary</i>
<i>Power-law</i>	$T = K \dot{\gamma}^n$	$\Psi = s^n$	<i>elementary</i>
<i>Sisko</i>	$T = \eta_\infty \dot{\gamma} + K \dot{\gamma}^n$	$\Psi = \mu s + \kappa s^n$	<i>elementary</i>
<i>Cross</i>	$\frac{\eta = \eta_\infty + \eta_0 - \eta_\infty}{1 + (\lambda \dot{\gamma})^m}$	$\Psi = s [\tilde{\eta}_\infty + \frac{\Delta}{1 + (\tilde{\lambda} s)^m}]$	<i>hypergeometric</i>
<i>Carreau-Yasuda</i>	<i>Eq. (67)</i>	$\Psi = s \tilde{\eta}(s)$	<i>hypergeometric</i>
<i>Ellis</i>	$\dot{\gamma} = \frac{\tau}{\eta_0} [1 + (\frac{\tau}{\tau_{1/2}})^{\alpha-1}]$	<i>naturally dual</i>	<i>elementary</i>
<i>Bingham-Papanastasiou</i>	<i>Eq. (117)</i>	$\Psi = \mu s + \beta(1 - e^{-Ms})$	<i>Lambert W</i>
<i>Herschel-Bulkley-Papanastasiou</i>	<i>Eq. (130)</i>	$\Psi = \alpha s^n + \beta(1 - e^{-Ms})$	<i>incomplete gamma</i>
<i>Ideal Herschel-Bulkley</i>	$T = \tau_y + K \dot{\gamma}^n$	$\Psi = \beta + (1 - \beta)s^n$ in sheared region	<i>nonsmooth conjugate</i>

As shown in Table 1, the special functions appearing in the final expressions are not fundamental to the method itself. They arise only when the normalized dissipation potential or the average-velocity moments are evaluated analytically for a particular model. The Carreau-Yasuda and Cross models lead to Gauss hypergeometric functions because their normalized potentials involve integrals of rational-power functions. The Cross model is included as a formal special case of the Carreau-Yasuda model, as discussed in Ref. [12], but it is listed separately because its simplified form is widely used. The Bingham–Papanastasiou model leads to exponentials and the Lambert W function because of the exponential regularization term. The Herschel-Bulkley-Papanastasiou model generally does not reduce to the standard Lambert W form for the wall shear rate when $n \neq 1$, but its average-velocity moments can be written using incomplete gamma functions. In contrast, Newtonian, power-law, and Sisko fluids lead to elementary formulas. Ellis fluids are naturally handled in the dual formulation because they are commonly specified by the inverse relation $\dot{\gamma} = \dot{\gamma}(\tau)$.

The ideal yield-stress limit highlights the main conceptual advantage of the Legendre-Fenchel viewpoint. Ideal Bingham and Herschel-Bulkley laws are nonsmooth at zero shear rate. In the present formulation, this nonsmoothness is not an obstacle. It is represented by the subdifferential of the dissipation potential. The corresponding complementary potential contains a flat interval, and this flat interval generates the plug region in capillary flow. Thus, the plug is not introduced as an additional geometric assumption. It follows directly from the dual constitutive structure.

The scope of this study is deliberately limited. The analysis is restricted to steady, fully developed, isothermal, pressure-driven flow of monotone generalized Newtonian fluids in geometries where the stress distribution is known. Within this setting, the Legendre-Fenchel framework gives a compact analytical structure for the velocity profile and the flow-rate relation. The purpose is not to replace numerical simulation or to cover all rheological effects, but to clarify the mathematical structure underlying a class of capillary-flow solutions.

Several extensions follow naturally. Wall slip can be incorporated by modifying the boundary condition, in which case the no-slip solution derived here becomes the bulk contribution and slip adds a stress-dependent velocity offset. Other geometries may also be treated when the stress distribution is available. Planar Poiseuille flow already fits within the same dimensionless formulation. Annular flow is more delicate because the stress field may change sign and the location of zero shear must be determined, but the same principle should apply once the appropriate stress distribution is specified.

The present formulation may also be useful for inverse capillary rheometry. Given pressure-drop/flow-rate data, one can fit the parameters of $T(\dot{\gamma})$ by matching the predicted $Q(G)$ relation. The separation between the wall shear-rate calculation, the normalized flow curve, and the velocity integration may help organize such inverse calculations.

7 Conclusion

In summary, the shear-rate parametrization used previously for the Carreau-Yasuda model is shown here to be a manifestation of Legendre-Fenchel conjugacy. The central result is that the velocity profile of a monotone generalized Newtonian fluid in capillary flow is generated by the complementary dissipation potential evaluated along the known stress distribution. This framework recovers the Carreau-Yasuda hypergeometric solution, explains the structure of several other generalized Newtonian models, and provides a natural treatment of ideal yield-stress limits through subdifferentials and flat regions of the dual potential.

Acknowledgments

This research is funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (Grant No. AP26197943).

References

- [1] Bird, R.B., Stewart, W.E., Lightfoot, E.N. *Transport Phenomena*. Revised 2nd ed. New York: John Wiley & Sons, 2006.
- [2] Chhabra, R.P., Richardson, J.F. *Non-Newtonian Flow and Applied Rheology: Engineering Applications*. 2nd ed. Amsterdam: Butterworth-Heinemann, 2008.
- [3] Buckingham, E. On plastic flow through capillary tubes // *Proceedings of the American Society for Testing Materials*. 1921. P. 1154–1156.
- [4] Mooney, M. Explicit formulas for slip and fluidity // *Journal of Rheology*. 1931. Vol. 2, No. 2. P. 210–222. <https://doi.org/10.1122/1.2116364>.
- [5] Meter, D.M., Bird, R.B. Tube flow of non-Newtonian polymer solutions: Part I. Laminar flow and rheological models // *AIChE Journal*. 1964. Vol. 10, No. 6. P. 878–881. <https://doi.org/10.1002/aic.690100619>.
- [6] Metzner, A.B., Reed, J.C. Flow of non-Newtonian fluids—correlation of the laminar, transition, and turbulent-flow regions // *AIChE Journal*. 1955. Vol. 1, No. 4. P. 434–440. <https://doi.org/10.1002/aic.690010409>.
- [7] Sochi, T. Using the Euler–Lagrange variational principle to obtain flow relations for generalized Newtonian fluids // *Rheologica Acta*. 2014. Vol. 53. P. 15–22. <https://doi.org/10.1007/s00397-013-0741-3>.
- [8] Sochi, T. Analytical solutions for the flow of Carreau and Cross fluids in circular pipes and thin slits // *Rheologica Acta*. 2015. Vol. 54. P. 745–756. <https://doi.org/10.1007/s00397-015-0863-x>.
- [9] Kim, S.K. Flow-rate based method for velocity of fully developed laminar flow in tubes // *Journal of Rheology*. 2018. Vol. 62, No. 6. P. 1397–1407. <https://doi.org/10.1122/1.5041958>.

- [10] Tozzi, E., Hartt, W. Non-Newtonian laminar flow in pipes using radius, stress, shear rate or velocity as the independent variable // *Physics of Fluids*. 2021. Vol. 33. Article 103104. <https://doi.org/10.1063/5.0067993>.
- [11] Boyko, E., Stone, H.A. Flow rate–pressure drop relation for shear-thinning fluids in narrow channels: approximate solutions and comparison with experiments // *Journal of Fluid Mechanics*. 2021. Vol. 923. Article A23. <https://doi.org/10.1017/jfm.2021.621>.
- [12] Wang, Y. Steady isothermal flow of a Carreau-Yasuda model fluid in a straight circular tube // *Journal of Non-Newtonian Fluid Mechanics*. 2022. Vol. 310. Article 104937. <https://doi.org/10.1016/j.jnnfm.2022.104937>.
- [13] Rockafellar, R.T. *Convex Analysis*. Princeton: Princeton University Press, 1970.
- [14] Boyd, S., Vandenberghe, L. *Convex Optimization*. Cambridge: Cambridge University Press, 2004.
- [15] Carreau, P.J. Rheological equations from molecular network theories // *Transactions of the Society of Rheology*. 1972. Vol. 16, No. 1. P. 99–127. <https://doi.org/10.1122/1.549276>.
- [16] Yasuda, K., Armstrong, R.C., Cohen, R.E. Shear flow properties of concentrated solutions of linear and star branched polystyrenes // *Rheologica Acta*. 1981. Vol. 20. P. 163–178. <https://doi.org/10.1007/BF01513059>.
- [17] Cross, M.M. Rheology of non-Newtonian fluids: a new flow equation for pseudoplastic systems // *Journal of Colloid Science*. 1965. Vol. 20, No. 5. P. 417–437. [https://doi.org/10.1016/0095-8522\(65\)90022-X](https://doi.org/10.1016/0095-8522(65)90022-X).
- [18] Papanastasiou, T.C. Flows of materials with yield // *Journal of Rheology*. 1987. Vol. 31, No. 5. P. 385–404. <https://doi.org/10.1122/1.549926>.
- [19] Herschel, W.H., Bulkley, R. Konsistenzmessungen von Gummi-Benzollösungen // *Kolloid-Zeitschrift*. 1926. Vol. 39. P. 291–300. <https://doi.org/10.1007/BF01432034>.
- [20] Frigaard, I. Simple yield stress fluids // *Current Opinion in Colloid & Interface Science*. 2019. Vol. 43. P. 80–93. <https://doi.org/10.1016/j.cocis.2019.03.002>.
- [21] Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (eds.). *NIST Handbook of Mathematical Functions*. New York: Cambridge University Press, 2010.
- [22] Corless, R.M., Gonnet, G.H., Hare, D.E.G., Jeffrey, D.J., Knuth, D.E. On the Lambert W function // *Advances in Computational Mathematics*. 1996. Vol. 5. P. 329–359. <https://doi.org/10.1007/BF02124750>.
- [23] You, Z., Huilgol, R.R., Mitsoulis, E. Application of the Lambert W function to steady shearing flows of the Papanastasiou model // *International Journal of Engineering Science*. 2008. Vol. 46, No. 8. P. 799–808. <https://doi.org/10.1016/j.ijengsci.2008.02.002>.
- [24] Pitsillou, R., Georgiou, G.C., Huilgol, R.R. On the use of the Lambert function in solving non-Newtonian flow problems // *Physics of Fluids*. 2020. Vol. 32, No. 9. <https://doi.org/10.1063/5.0021437>.
- [25] Housiadas, K.D., Georgiou, G.C. The analytical solution for the flow of a Papanastasiou fluid in ducts with variable geometry // *Journal of Non-Newtonian Fluid Mechanics*. 2023. Vol. 319. Article 105074. <https://doi.org/10.1016/j.jnnfm.2023.105074>.